Harnack inequality for hypoelliptic kinetic equations with bounded measurable coefficients

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joint work with Golse, Mouhot, Vasseur

Mathematical topics in kinetic theory

Cambridge, May 9-13, 2016

Outline

1. Kinetic Fokker-Planck equations with rough coefficients
2. The $L^2 - L^\infty$ bound (following Moser)
3. Improvement of oscillation (following de Giorgi)

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A model from kinetic theory

A non-linear Fokker-Planck equation

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= \rho[f] \; \nabla_v \big(\nabla_v f + v f \big) \\ \rho[f] &:= \int f(t,x,v) \mathrm{d}v \end{aligned}$$

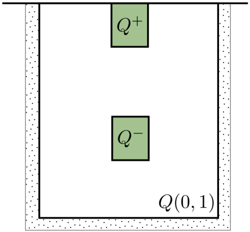
- Original problem with C. Mouhot
- First attempt: hypo-coercivity approach // energy estimate
- A control of the modulus of continuity is needed (?)

Harnack inequality

(Kinetic Fokker-Planck equation)

$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f)$ in $Q(0,1)$

$A(t,x,v)$ symmetric
eigenvalues of A in $[\lambda, \Lambda]$ with $\lambda > 0$



Theorem (Golse-Cl-Mouhot-Vasseur)

Then any non-negative weak solution f in $Q(0,1)$ satisfies

$$\sup_{Q^-} f \leq C \inf_{Q^+} f.$$

Harnack inequality

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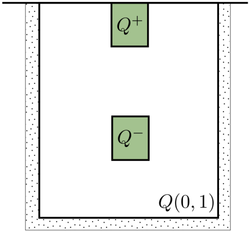

- Boundedness of solutions by Pascucci-Polidoro (2004)

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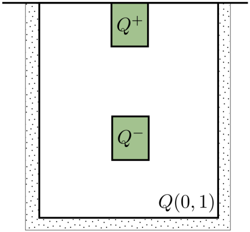

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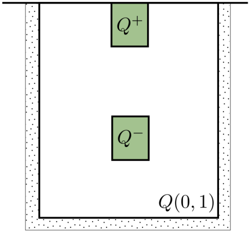

- Boundedness of solutions by Pascucci-Polidoro (2004)
- **Hölder continuity of solutions by Wang & Zhang (2009)**
- Decisive interaction with Luis Silvestre

Elliptic regularity: Hilbert's 19^{th} problem

Given F smooth & strictly convex
Show that minimizers of $\int F(\nabla w)$ are analytic

- Euler-Lagrange equation: $\nabla \cdot (\nabla F(\nabla w)) = 0$
- Differentiate the equation: $\left| \begin{array}{l} u = \partial_i w \text{ solves } \nabla \cdot (A \nabla u) = 0 \\ \text{with } A_{ij} = \partial_{i,j} F(\nabla w) \end{array} \right.$
- Schauder: $w \in C^{1,\alpha} \Rightarrow A \in C^{0,\alpha} \Rightarrow u$ is smooth
- Classical theory: w smooth $\Rightarrow w$ is analytic

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How to reach $w \in C^{1,\alpha}$?

Elliptic regularity: Hilbert's 19$^{\text{th}}$ problem

$$\nabla(A\nabla u) = 0 \quad \text{in } B_1$$

How to reach $u \in C^{0,\alpha}$?

Elliptic regularity: De Giorgi-Nash-Moser

$$\nabla(A\nabla u) = 0 \quad \text{in } B_1$$

How to reach $u \in C^{0,\alpha}$?

- De Giorgi (1956) proves an $L^2 - L^\infty$ **bound** and shows that it implies the decrease of oscillation thanks to an **isoperimetric lemma**
- Nash (1958) used estimates on the fundamental solution
- Moser (1961) gets the $L^2 - L^\infty$ bound through an **iteration procedure** and reaches a Harnack inequality by relating positive and negative Lebesgue norms (through the study of $\ln u$)

Hypoelliptic equations

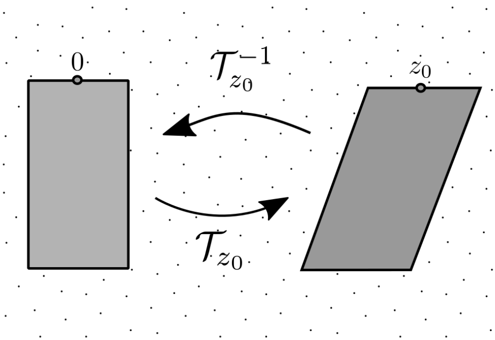
The case $A \equiv Id$: $$\partial_t f + v \cdot \nabla_x f = \Delta_v f$$

- Kolmogoroff (1934):
  - Explicit fundamental solution
  - The equation has a regularizing effect
- Hörmander (1967):
  - Starting point for hypoelliptic theory
  - Commutator estimates

Scaling, transformation and cylinders

Scaling: $f(r^2 t, r^3 x, r v)$ satisfies the same equation for any $r > 0$

Transformation: $\mathcal{T}_{z_0}(z) = (t_0 + t, x_0 + x + t v_0, v_0 + v)$



Cylinders: $Q(0, r) = (-r^2, 0] \times B(0, r^3) \times B(0, r)$

Convex change of unknown and energy

Convex function φ: if f is a solution then $\varphi(f)$ is a subsolution

- $\varphi(f) = \max(f, 0) = f_+$
- $\varphi(f) = f_+^q$ with $q \geq 1$

Central energy estimate:

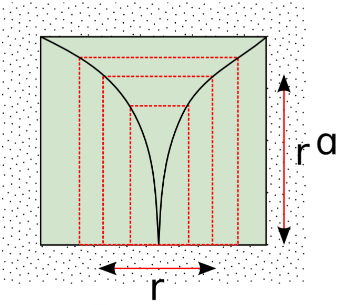
$$\int f^2 \Psi^2|_{t=t_1} + \lambda \int |\nabla_v f|^2 \Psi^2 \leq C \int_{Q_0} f^2$$

for Ψ supported in Q_0 and vanishing at initial time

Towards Hölder regularity / Harnack inequality

Hölder regularity

Get $\boxed{\operatorname*{osc}_{Q(0,r)} f \le Cr^{\alpha}}$ with $C > 0$, $\alpha \in (0,1)$



It is enough to get $\boxed{\operatorname*{osc}_{Q(0,2^{-k})} f \le \theta^k \operatorname*{osc}_{Q(0,1)} f}$ with $\theta \in (0,1)$

How to reach the algebraic decay of the oscillation

- Prove that L^2 solutions are in fact L^p for $p > 2$
- Iterate this result to get that they are in fact L^∞
- Get decrease of oscillation from a large box to a smaller one

$$\boxed{\operatorname*{osc}_{Q(0,1/2)} f \le \theta \operatorname*{osc}_{Q(0,1)} f} \text{ with } \theta \in (0,1)$$

Harnack inequality

- The **decrease of oscillation** is needed ...
- ... combined with "**the propagation of minima**"

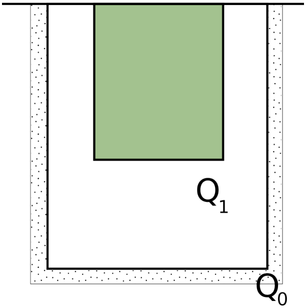
Outline

The $L^2 - L^\infty$ bound

Theorem (Pascucci-Polidoro // GIMV)

$$\partial_t f + v \cdot \nabla_x f = \nabla_v (A \nabla_v f) \quad \textit{in } Q_0$$
$$\Rightarrow \|f^+\|_{L^\infty(Q_1)} \leq C \|f^+\|_{L^2(Q_0)}$$

where $C = C(d, \lambda, \Lambda, Q_0, Q_1)$



Sketch of the proof (Moser's iteration)

- f^+ is a sub-solution

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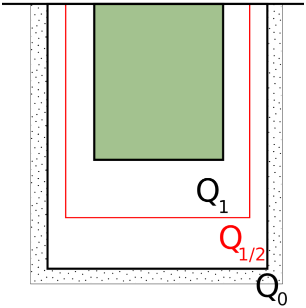

Sketch of the proof (Moser's iteration)

- f^+ is a sub-solution
- Find $p > 2$ universal s.t.

$$\left. \begin{array}{l} f \ \textit{sub}\text{-solution} \\ f \geq 0 \end{array} \right| \Rightarrow \|f\|_{L^p(Q_{\frac{1}{2}})} \leq C_{0,\frac{1}{2}} \|f\|_{L^2(Q_0)}$$

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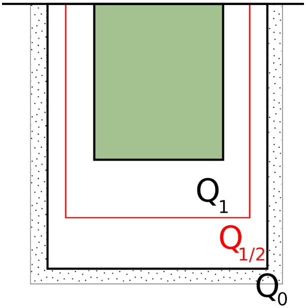

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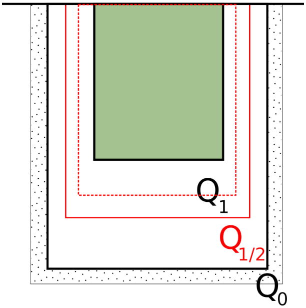

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The $L^2 - L^\infty$ bound

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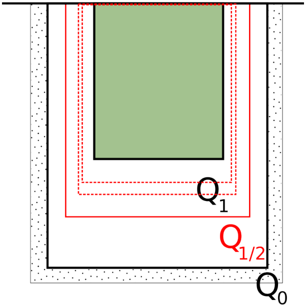

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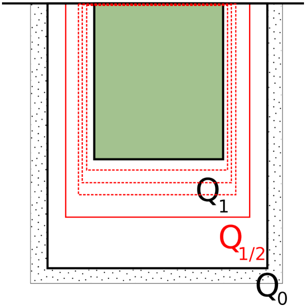

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Local gain of integrability

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Assume first that f is a solution

- Recall the central energy estimate:

$$\lambda \int |\nabla_v f|^2 \Psi^2 \qquad \leq C \int_{Q_0} f^2$$

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$$\lambda \int |\nabla_v f|^2 \Psi^2 + \|D_x^{\frac{1}{3}} f\|^2_{L^2(Q_{\frac{1}{2}})} + \|D_t^{\frac{1}{3}} f\|^2_{L^2(Q_{\frac{1}{2}})} \leq C \int_{Q_0} f^2$$

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Assume now that f is a sub-solution

- Associate with sub-sol'n F the sol'n $\tilde{F}$ of the transport eq'n
- Get improvement of regularity for $\tilde{F}$
- Use comparison principle to transfer gain of int. from $\tilde{F}$ to F

Averaging lemma / transfer of regularity

Agoshkov, Golse-Perthame-Sentis ('84,'85)
Transfer of regularity for (v) variable to (x,t) variables.

A result by Bouchut (2002)
Given $F \in L^2$ with $D_v F \in L^2$ satisfying

$$\boxed{\partial_t F + v \cdot \nabla_x F = \nabla_v \cdot g_1 + g_0}$$

we have

$$\|D_t^{\frac{1}{3}} F\|_2 + \|D_x^{\frac{1}{3}} F\|_2 \leq \|F\|_2 + \|D_v F\|_2 + \|g_0\|_2 + \|g_1\|_2$$

Application
Take $F = f\Psi$ where $\Psi =$ cut-off of the central energy estimate

$$\lambda \int |\nabla_v f|^2 \Psi^2 + \|D_x^{\frac{1}{3}} f\|^2_{L^2(Q_1)} + \|D_t^{\frac{1}{3}} f\|^2_{L^2(Q_1)} \leq C \int_{Q_0} f^2$$

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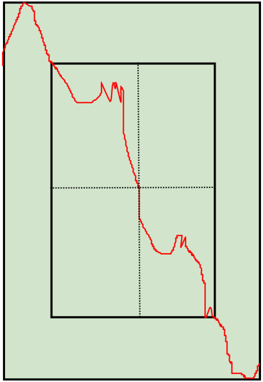
Improvement of oscillation



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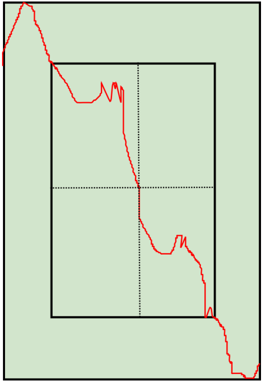

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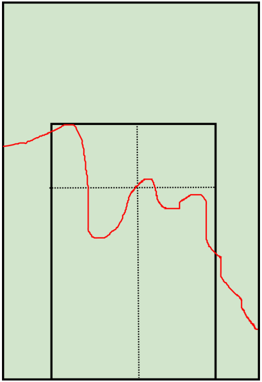

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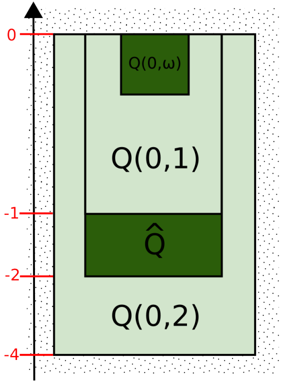
  - If f lies "more above 0 than below", then the infimum increases

Decrease of the upper bound

Assume $\boxed{\begin{array}{c} |f| \leq 1 \text{ in } Q(0,2) \\ \hline |\{f \leq 0\} \cap \hat{Q}| \geq \frac{1}{2}|\hat{Q}| \end{array}}$

Find $\lambda \in (0,1)$ s.t. $\boxed{f \leq 1 - \lambda \text{ in } Q(0, \omega/2)}$



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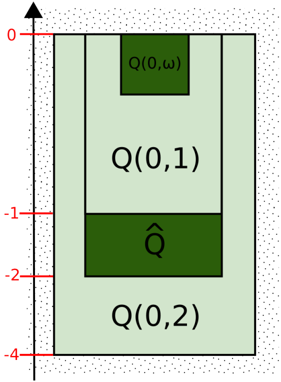
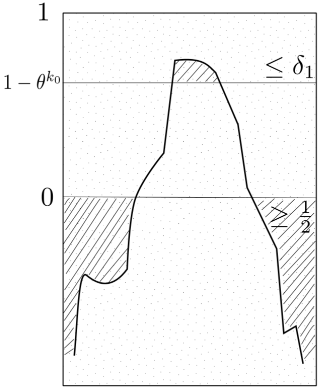


Information in measure translates into a pointwise estimate

Truncation

Define $f_{k_0} = \theta^{-k_0}(f - (1 - \theta^{k_0}))$

$\boxed{\text{Find } k_0 : |\{f_{k_0} \geq 0\} \cap Q(0,\omega)| \leq \delta_1}$



Consequence

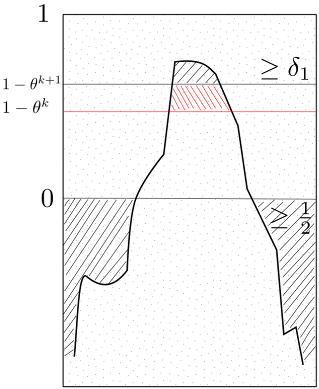
- $\|f_{k_0}^+\|_{L^\infty(Q(0,\omega/2))} \leq C\|f_{k_0}^+\|_{L^2(Q(0,\omega))} \leq C\sqrt{\delta_1} \leq 1 - \theta$
- Finally, $\boxed{f \leq} 1 - \theta^{k_0+1} = \boxed{1 - \lambda}$

Iterative truncation

$\boxed{\text{If } |\{f_{k+1} \geq 0\}| \geq \delta_1}$

$$\begin{aligned} f_k &= \theta^{-k}(f - (1 - \theta^k)) \\ f_{k+1} &= \theta^{-1}(f_k - (1 - \theta)) \end{aligned}$$

$$\begin{aligned} \{f_{k+1} \geq 0\} &= \{f_k \geq 1 - \theta\} \\ \{f_{k+1} \leq 0\} &\supset \{f_k \leq 0\} \end{aligned}$$



$\boxed{\text{Then}}$

$$|\{f_k \geq 1 - \theta| \geq \delta_1$$
$$|\{f_k \leq 0\}| \geq \delta_2$$

A hypoelliptic intermediate-value lemma (à la de Giorgi)

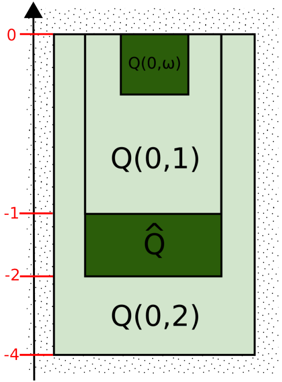
Theorem (GIMV)

for all $f \leq 1$ **solution in** $Q(0,2)$ *such that*

$$\begin{cases} |\{f \geq 1-\theta\} \cap Q(0,\omega)| \geq \delta_1 \\ |\{f \leq 0\} \cap \hat{Q}| \geq \delta_2 \end{cases}$$

then

$$|\{0 < f < 1-\theta\} \cap Q(0,1) \cup \hat{Q}| \geq \alpha$$



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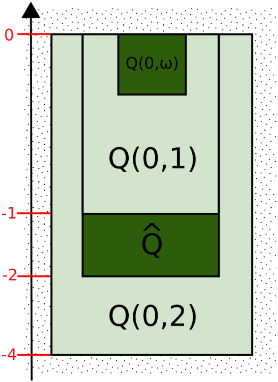


De Giorgi's classical argument: if an H^1 function takes values above $1-\theta$ and below 0, intermediate values "occupy some non-negligeable space"

A hypoelliptic intermediate-value lemma (à la de Giorgi)

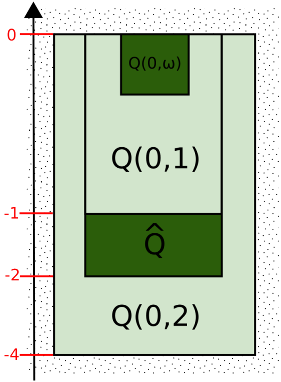
Theorem (GIMV)

for all $f \leq 1$ **solution in** $Q(0,2)$ *such that*

$$\begin{cases} |\{f \geq 1-\theta\} \cap Q(0,\omega)| \geq \delta_1 \\ |\{f \leq 0\} \cap \hat{Q}| \geq \delta_2 \end{cases}$$

then

$$|\{0 < f < 1-\theta\} \cap Q(0,1) \cup \hat{Q}| \geq \alpha$$



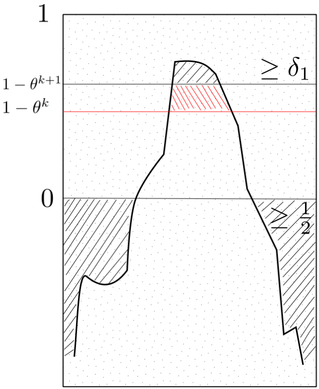
De Giorgi's classical argument: if an H^1 function takes values above $1-\theta$ and below 0, intermediate values "occupy some non-negligeable space" **(quantitative version)**

Iterative truncation (again)

If $|\{f_{k+1} \geq 0\}| \geq \delta_1$

$$\begin{aligned} f_k &= \theta^{-k}(f - (1-\theta^k)) \\ f_{k+1} &= \theta^{-1}(f_k - (1-\theta)) \end{aligned}$$

$$\begin{aligned} \{f_{k+1} \geq 0\} &= \{f_k \geq 1-\theta\} \\ \{f_{k+1} \leq 0\} &\supset \{f_k \leq 0\} \end{aligned}$$



Then

$$\begin{aligned} |\{f_k \geq 1-\theta| &\geq \delta_1 \\ |\{f_k \leq 0\}| &\geq \delta_2 \end{aligned}$$

Iterative truncation (again)

If $|\{f_{k+1} \geq 0\}| \geq \delta_1$

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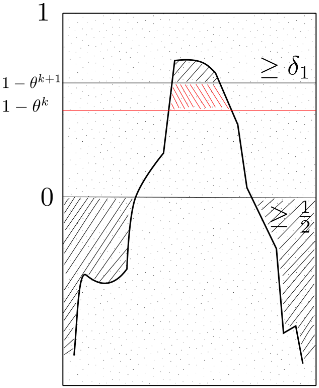


Then

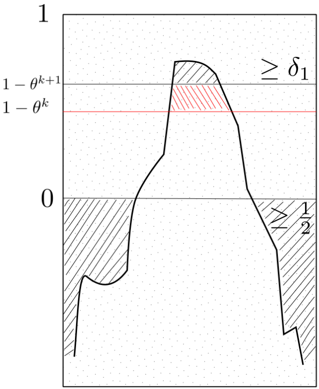
$$\alpha \leq |\{0 < f_k < 1-\theta\}|$$

Iterative truncation (again)

If $|\{f_{k+1} \geq 0\}| \geq \delta_1$

$$
\begin{aligned}
f_k &= \theta^{-k}(f - (1 - \theta^k)) \\
f_{k+1} &= \theta^{-1}(f_k - (1 - \theta))
\end{aligned}
$$

$$
\begin{aligned}
\{f_{k+1} \geq 0\} &= \{f_k \geq 1 - \theta\} \\
\{f_{k+1} \leq 0\} &\supset \{f_k \leq 0\}
\end{aligned}
$$

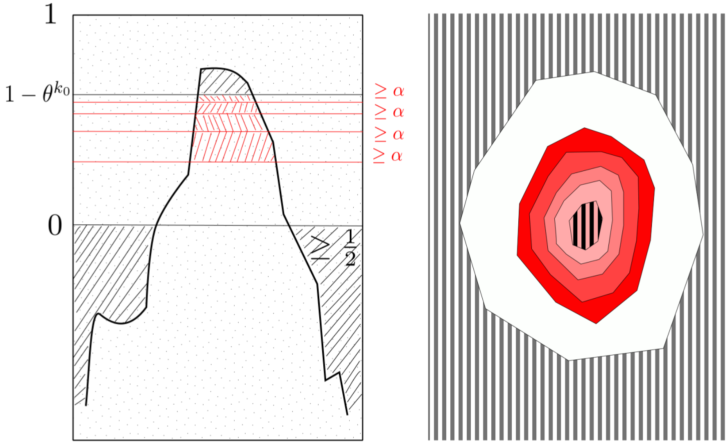


Then

$$\alpha \leq |\{1 - \theta^k < f < 1 - \theta^{k+1}\}|$$

Iterated truncation (again and again)

Graph and level sets



The intermediate-value lemma: sketch of the proof

Proof by contradiction

- Solutions $f_k \leq 1$, matrices A_k, $\theta_k \to 0$

$$
\begin{aligned}
|\{f_k \geq 1-\theta_k\} \cap Q(0,\omega)| &\geq \delta_1 \\
|\{f_k \leq 0 \cap \hat{Q}| &\geq \delta_2 \\
|\{0 < f_k < 1-\theta_k\} \cap Q(0,1) \cup \hat{Q}| &\to 0
\end{aligned}
$$

- $F_k = f_k^+ = \max(f_k, 0) \in [0,1]$ is a sub-solution
- Refined averaging lemma $\Rightarrow$ to pass to the limit
  $F_k \to F \in [0,1]$ with

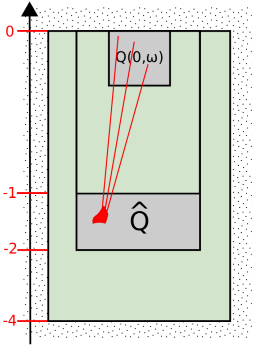
$$
\begin{aligned}
|\{F = 1\} \cap Q(0,\omega)| &\geq \delta_1 \\
|\{F = 0 \cap \hat{Q}| &\geq \delta_2 \\
|\{0 < F < 1\} \cap Q(0,1) \cup \hat{Q}| &= 0.
\end{aligned}
$$

The intermediate-value lemma: sketch of the proof

$F(t, x, v) = \mathbf{1}_P(t, x)$ with

$$\boxed{\partial_t F + v \cdot \nabla_x F \leq 0 \text{ with } t \in (-2, 0)}$$

$$|P \cap Q(0, \omega)| \geq \delta_1$$
$$|\hat{Q} \setminus P| \geq \delta_2$$



- Study the **propagation of** $\{F = 0\}$
- reach a contradiction

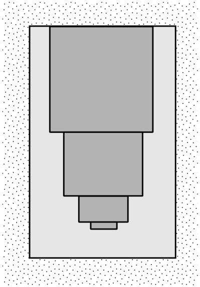
Harnack inequality

Two ingredients

- **Decrease of oscillation:**

$$\boxed{|f| \leq 1} \text{ in } Q(0,2) \quad \Rightarrow \quad \boxed{\operatorname{osc} f \leq 1-\theta} \text{ in } Q(0,\omega)$$

- **Propagation of minima:**



$$\forall z \in Q^-, \qquad \min_{Q^+} f \gtrsim r^q \min_{Q(z,r)} f$$

Related results - perspectives

Related results

- Improved integrability of $\nabla_v f$
- Regularity of sub-solutions
- Source terms

Under progress - perspectives

- Quantitative version of the intermediate-value lemma
- Consequence for the **Landau equation**
- Integro-differential setting ... towards **Boltzmann equation**

References

- **Cl-Mouhot**, *Hölder continuity of solutions to hypoelliptic equations with bounded measurable coefficients*, arXiv 1505.04608
- **Golse-Vasseur**, *Hölder regularity for hypoelliptic kinetic equations with rough diffusion coefficients*, arXiv 1506.01908
- **Golse-Cl-Mouhot-Vasseur**, *Hölder regularity for hypoelliptic kinetic equations with bounded measurable coefficients*, in preparation

References

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Thank you for your attention