From Newton's law to the linear Boltzmann equation without cut-off

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13 Mai 2016

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$$x = \text{position}$$
$$v = \text{velocity}$$

Historical results :

- A fundamental example, **the Boltzmann equation** (1872) = the evolution equation for the density of particles of a sufficiently rarefied gas.

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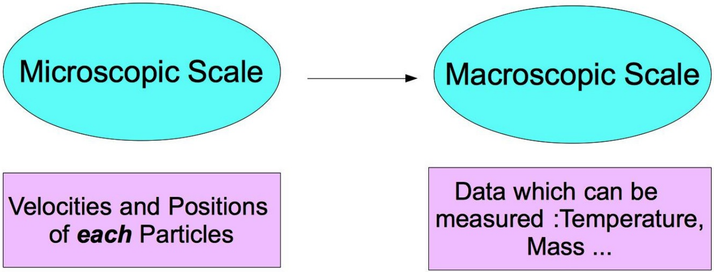

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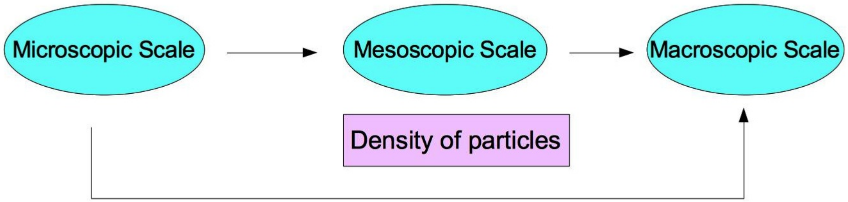

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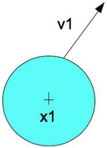

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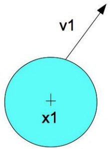
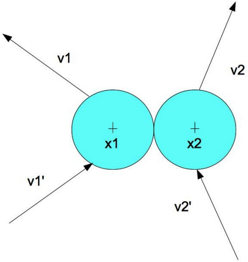
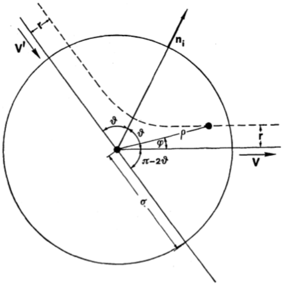


Particles bounces off according to the laws of **elastic reflection**.



Recently rigorously proved by *Gallagher, Saint-Raymond and Texier*, *Pulvirenti, Saffirio and Simonella* in the context of **hard-spheres** and **short range potentials**.



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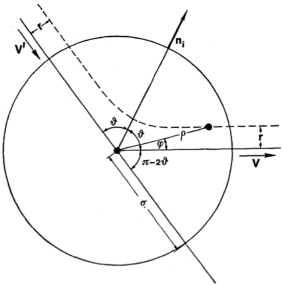

(Figure : *The Boltzmann equation and its application, Cercignani.*)

Our context : **infinite-range potentials**.

The Boltzmann equation without cut-off

$$\partial_t f + v.\nabla_x f = Q(f,f)$$

where $Q(f,f) = \int\int [f'f'_1 - ff_1]B(v - v_1, \omega)dv_1 d\omega$, with $f = f(v)$, $f' = f(v')$, $f_1 = f(v_1)$, $f'_1 = f(v'_1)$.

B = **cross-section**, $B(|v - v_1|, cos\theta)$ with θ = **deviation angle**.

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B = **cross-section**, $B(|v - v_1|, cos\theta)$ with θ = **deviation angle**.

Example

Inverse-power law potentials :

$$\Phi(r) = \frac{1}{r^{s-1}},\ s > 2.$$

The **cross-section** satisfies

$$B(|v - v_1|, \cos\theta) = b(cos\theta)|v - v_1|^\gamma,$$

$$sin^{d-2}\theta b(\cos\theta) \sim C\theta^{-1-\alpha},\ \alpha > 0$$

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$$f(v)f(v_1) \approx f(v')f(v'_1)$$

Result

Microscopic Model : for $i \in \{1, \dots, N\}$, $x_i \in \mathbf{T}^d$, $v_i \in \mathbf{R}^d$,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = -\dfrac{1}{\varepsilon} \displaystyle\sum_{j \neq i} \nabla \Phi\left(\dfrac{x_i - x_j}{\varepsilon}\right). \end{cases}$$

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Small perturbation around the equilibrium :

$$f_N^0(Z_N) := M_{N,\beta}(Z_N)\rho^0(x_1) \tag{1}$$

ρ^0 **density of probability** on $\mathbf{T}^d$, $M_{N,\beta}$ **Gibbs measure** : for $\beta > 0$

$$M_{N,\beta}(Z_N) := \frac{1}{\mathcal{Z}_N} \left(\frac{\beta}{2\pi}\right)^{dN/2} \exp(-\beta H_N(Z_N)) \tag{2}$$

with $H_N(Z_N) := \displaystyle\sum_{1 \leq i \leq N} \frac{1}{2}|v_i|^2 + \sum_{1 \leq i < j \leq N} \Phi\left(\frac{(x_i - x_j)}{\varepsilon}\right).$

Theorem (A., 2016)

*Consider **the initial distribution** f_N^0 defined above, then the **distribution** $f_N^{(1)}(t,x,v)$ **of the tagged particle** converges in $\mathcal{D}'(\mathbf{T}^d \times \mathbf{R}^d)$ when N goes to ∞ **under the Boltzmann-Grad scaling** $N\varepsilon^{d-1} = 1$ to $M_\beta(v)h(t,x,v)$ where $h(t,x,v)$ is the solution of **the linear Boltzmann equation without cut-off***

$$\partial_t h + v.\nabla_x h = -\int\int [h(v) - h(v_1)] M_\beta(v_1) b(v - v_1) dv_1 d\nu \qquad (3)$$

with initial data $\rho^0(x_1)$ and where $M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{d/2} \exp\left(-\frac{\beta}{2}|v|^2\right)$, $\beta > 0$.

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Idea : $\nabla\Phi = \nabla\Phi_{>R} + \nabla\Phi_{<R}$.

The hard-sphere case

The **BBGKY hierarchy** : for $s < N$,

$$(\partial_t + \sum_{i=1}^{s} v_i.\nabla_{x_i}) f_N^{(s)}(t, Z_s) = (\mathcal{C}_{s,s+1} f_N^{(s+1)})(t, Z_s)$$

with $\mathcal{C}_{s,s+1}$ = **the collision term**, $Z_s = (z_1, \ldots, z_s)$, $z_i = (x_i, v_i)$.

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Duhamel's formula :

$$f_N^{(s)}(t) = \mathcal{T}_s(t) f_N^{(s)}(0) + \int_0^t \mathcal{T}_s(t - t_1) \mathcal{C}_{s,s+1} f_N^{(s+1)}(t_1) dt_1.$$

Iterated Duhamel's formula.

The BBGKY series :

$$f_N^{(s)}(t) = \sum_{n=0}^{N-s} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathcal{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathcal{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots\mathcal{T}_{s+n}(t_n) f_N^{(s+n)}(0)\,dt_n\dots dt_1.$$

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$$g^{(s)}(t) = \sum_{n\geq 0} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathcal{T}_s^0(t-t_1)\mathcal{C}_{s,s+1}^0\mathcal{T}_{s+1}^0(t_1-t_2)\mathcal{C}_{s+1,s+2}^0\dots\mathcal{T}_{s+n}(t_n) g^{(s+n)}(0)dt_n\dots dt_1$$

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Strategy : Notion of **pseudo-trajectories.**

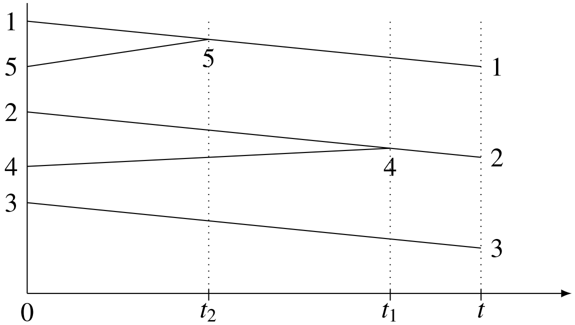
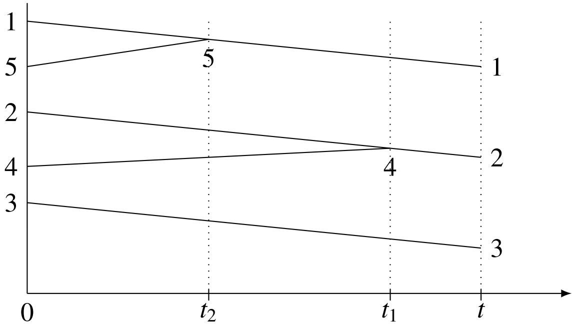


Figure : Representation of **a collision tree** associated to the term $\int_0^t \int_0^{t_1} \mathcal{T}_3^0(t-t_1)\mathcal{C}_{3,4}^{j_1,2}\mathcal{T}_4^0(t_1-t_2)\mathcal{C}_{4,5}^{j_2,1}\mathcal{T}_5(t_2)f_N^{(5)}(0)dt_2dt_1$.



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Strategy : **Coupling** of the pseudo-trajectories.

Recollision

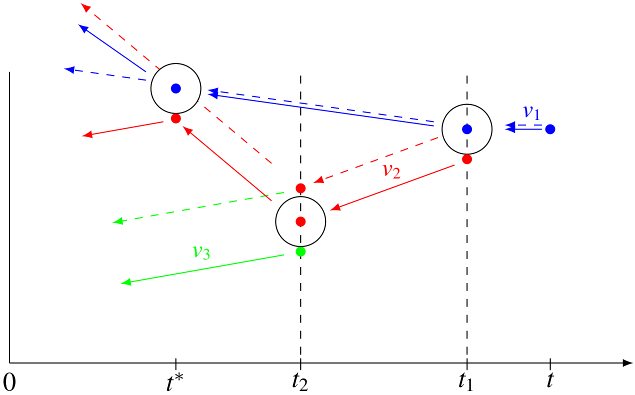

Figure : An example of a recollision between particles 1 and 2 at time t^*.

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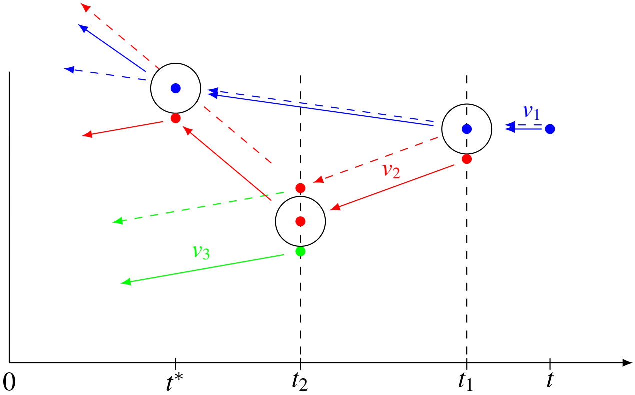

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Strategy : **Geometrical control** of the recollisions

The linear Boltzmann equation case

Bodineau, Gallagher and Saint-Raymond (2014) : **overcome the difficulty of the short time validity** in the case of a fluctuation around the equilibrium.

The linear Boltzmann equation without cut-off

Truncated marginals :

$$\tilde{f}_{N,R}^{(s)}(t, Z_s) := \int_{\mathbf{T}^{d(N-s)} \times \mathbf{R}^{d(N-s)}} f_N(t, Z_s, z_{s+1}, \dots, z_N) \prod_{\substack{1 \le i \le s \\ s+1 \le j \le N}} \mathbf{1}_{\{|x_i - x_j| > R\varepsilon\}} dz_{s+1} \dots dz_N$$

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The **BBGKY hierachy** : for $s<N$,

$$\partial_t \tilde{f}^{(s)}_{N,R} + \sum_{i=1}^{s} v_i.\nabla_{x_i}\tilde{f}^{(s)}_{N,R} - \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s} \nabla\Phi^{<}\left(\frac{x_i-x_j}{\varepsilon}\right).\nabla_{v_i}\tilde{f}^{(s)}_{N,R}$$

$$= \mathcal{C}_{s,s+1}\tilde{f}^{(s+1)}_{N,R} + \mathcal{C}_{s,s+1}\bar{f}^{(s+1)}_{N,R} + \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s} \nabla\Phi^{>}\left(\frac{x_i-x_j}{\varepsilon}\right).\nabla_{v_i}\tilde{f}^{(s)}_{N,R}$$

$$+ \frac{(N-s)}{\varepsilon}\sum_{i=1}^{s}\int_{\mathbf{T}^{d(N-s)}\times\mathbf{R}^{d(N-s)}} \nabla\Phi\left(\frac{x_i-x_{s+1}}{\varepsilon}\right).\nabla_{v_i} f_N(t,Z_N)$$

$$\prod_{\substack{1\le l\le s\\ s+1\le k\le N}} \mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}} dZ_{(s+1,N)}$$

Duhamel's formula :

$$
\begin{aligned}
\tilde{f}^{(s)}_{N,R}(t,Z_s) = \mathcal{S}_s(t)\tilde{f}^{(s)}_{N,R}(0,Z_s) \\
&+\int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\tilde{f}^{(s+1)}_{N,R}(t_1,Z_s)dt_1 \\
&+\int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\bar{f}^{(s+1)}_{N,R}(t_1,Z_s)dt_1 \\
&+\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s}\int_0^t \mathcal{S}_s(t-t_1)\left[\nabla\Phi^{>}\left(\frac{x_i-x_j}{\varepsilon}\right).\nabla_{v_i}\tilde{f}^{(s)}_{N,R}\right](t_1,Z_s)dt_1 \\
&+\frac{(N-s)}{\varepsilon}\sum_{i=1}^{s}\int_0^t \mathcal{S}_s(t-t_1)\left[\int_{\mathbf{T}^{d(N-s)}\times\mathbf{R}^{d(N-s)}}\nabla\Phi\left(\frac{x_i-x_{s+1}}{\varepsilon}\right).\nabla_{v_i}f_N \right. \\
&\qquad \left. \prod_{\substack{1\leq l\leq s\\ s+1\leq k\leq N}}\mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}}dZ_{(s+1,N)}\right](t_1,Z_s)dt_1
\end{aligned}
$$

Obstacles to the convergence

Four **possible obstacles** to the convergence :

- the **very long-range** interactions,
- **multiple simultaneous** interactions,
- the presence of **recollisions**,
- **super-exponential** collision process.

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New terms $\Longrightarrow$ **Weak approach.**

Strategy

Iteration on the term :

$$\int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\tilde{f}_{N,R}^{(s+1)}(t_1, Z_s)dt_1.$$

Definition of the **operators** :

$$Q_{s,s}(t) := \mathcal{S}_s(t)$$

$$Q_{s,s+n}(t) := \int_0^{t-\delta}\int_0^{t_1-\delta}\dots\int_0^{t_{n-1}-\delta} \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\chi_{H_{s+1}}\left(1-\chi_{geom(s+1)}\right)\chi_{\eta_s+1}\dots$$
$$\dots\mathcal{S}_{s+n-1}(t_{n-1}-t_n)\mathcal{C}_{s+n-1,s+n}\chi_{H_{s+n}}\left(1-\chi_{geom(s+n)}\right)\chi_{\eta_{s+n}}\mathcal{S}_{s+n}(t_n)dt_n\dots dt_1.$$

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Remainders associated to the **long-range part** :

$$r^{Pot,a}_{s,m+1}(0,t,Z_s) := \sum_{n=0}^{m}\int_0^{t-n\delta} Q_{s,s+n}(t-t_{n+1})$$

$$\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s+n}\left[\nabla\Phi^{>}\left(\frac{x_i-x_j}{\varepsilon}\right).\nabla_{v_i}\tilde{f}^{(s+n)}_{N,R}\right](t_{n+1},Z_s)dt_{n+1}$$

$$r^{Pot,b}_{s,m+1}(0,t,Z_s) := \sum_{n=0}^{m} \int_0^{t-n\delta} Q_{s,s+n}(t-t_{n+1})$$

$$\frac{(N-(s+n))}{\varepsilon} \sum_{i=1}^{s+n} \Bigg[\int_{\mathbf{T}^{d(N-(s+n))}\times\mathbf{R}^{d(N-(s+n))}} \nabla\Phi(\frac{x_i - x_{s+n+1}}{\varepsilon}).\nabla_{v_i} f_N$$

$$\prod_{\substack{1\le l\le s+n \\ s+n+1\le k\le N}} \mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}} dZ_{(s+n,N)} \Bigg] (t_{n+1}, Z_s) dt_{n+1}.$$

Advantage of the iteration method

no pathological situations $\Rightarrow$ easy to pass from $\mathbf{Z_m}(\mathbf{t_m})$ to $\mathbf{z}$ (state of particle 1 at time t) via **changes of variables** such as

$$\begin{array}{rcl} \mathbf{T}^d \times \mathbf{R}^d \times [0, t-\delta] \times \mathbf{S}^{d-1} \times \mathbf{R}^d \times \cdots \times [0, t_{m-1}-\delta] \times \mathbf{S}^{d-1} \times \mathbf{R}^d & \to & \mathbf{T}^{(m+1)d} \times \mathbf{R}^{(m+1)d} \\ (z, t_1, \nu_2, v_2, \ldots, t_m, \nu_{m+1}, v_{m+1}) & \mapsto & \tilde{Z}_{m+1} = Z_{m+1}(t_m). \end{array}$$

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Key point : z Lipschitz function of $(\tilde{x}_1, \tilde{v}_1, \ldots, \tilde{x}_{m+1}, \tilde{v}_{m+1})$

Advantage of the iteration method

no pathological situations $\Rightarrow$ easy to pass from $\mathbf{Z_m}(\mathbf{t_m})$ to $\mathbf{z}$ (state of particle 1 at time t) via **changes of variables** such as

$$\begin{array}{rcl} \mathbf{T}^d \times \mathbf{R}^d \times [0, t-\delta] \times \mathbf{S}^{d-1} \times \mathbf{R}^d \times \cdots \times [0, t_{m-1}-\delta] \times \mathbf{S}^{d-1} \times \mathbf{R}^d & \to & \mathbf{T}^{(m+1)d} \times \mathbf{R}^{(m+1)d} \\ (z, t_1, \nu_2, v_2, \ldots, t_m, \nu_{m+1}, v_{m+1}) & \mapsto & \tilde{Z}_{m+1} = Z_{m+1}(t_m). \end{array}$$

Key point : z Lipschitz function of $(\tilde{x}_1, \tilde{v}_1, \ldots, \tilde{x}_{m+1}, \tilde{v}_{m+1})$

Tools to prove it : study of the reduced dynamics

- **bound on the microscopic time of interaction** thanks to the **lower bound on relative velocites**,
- use of **the Cauchy-Lipschitz theorem**, $\nabla\Phi$ being Lipschitz,
- **Lipschitz character** of the collision times.

Lipschitz control associated to the pseudo-trajectories $\Longrightarrow$ bound of the remainder associated to the long-range part controlled by

$$\left(e^{C\frac{R^2}{\eta}}\right)^{2^{K+1}} \|\nabla\Phi^>\|_\infty .$$

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Limit of the approach : Very decreasing potentials.

Thank you for your attention.