Mathematical modelling of collective behavior

Young-Pil Choi

Fakultät für Mathematik
Technische Universität München

This talk is based on joint works with José A. Carrillo, Maxime Hauray, and Samir Salem

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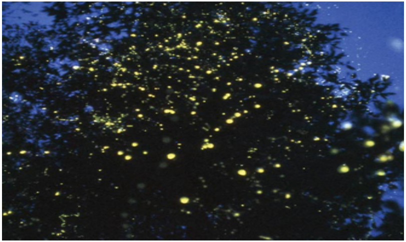
Swarming in Nature




Individual Based Models

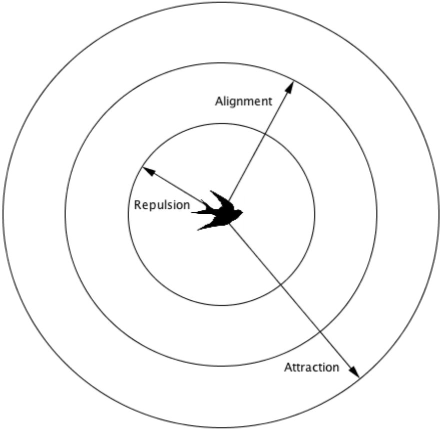


Figure: 3 Zones

Cucker-Smale model

Let (x_i, v_i) be the position and velocity of i-th particle. Then Cucker-Smale model reads as

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \; i = 1, \cdots, N,$$
$$\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} \psi(x_j - x_i)(v_j - v_i), \quad \psi(x) := \frac{1}{(1+|x|^2)^{\beta/2}}, \quad \beta \geq 0.$$

Definition: Let $\mathcal{P} := \{(x_i, v_i)\}_{i=1}^N$ be an N-body interacting system. Then $\mathcal{P}$ exhibits a flocking if and only if the following two relations hold:

$$\lim_{t\to\infty} |v_i(t) - v_j(t)| = 0 \quad \text{and} \quad \sup_{0\leq t<\infty} |x_i(t) - x_j(t)| < \infty, \quad 1 \leq i, j \leq N.$$

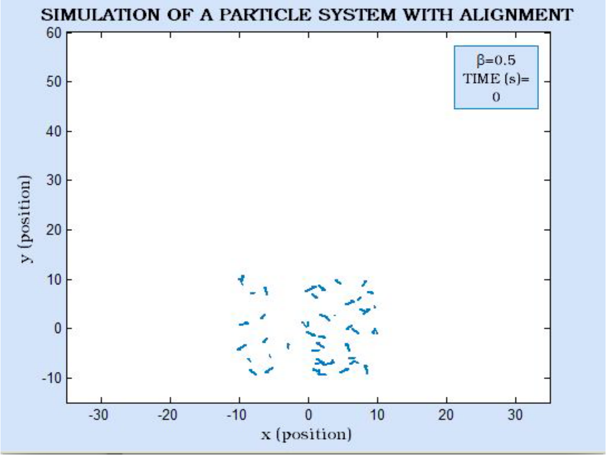
Property: Conservation of momentum.

Flocking estimate: If $\int^\infty \psi(r)\, dr = \infty \; (\iff \beta \leq 1)$, then we have

$$\sup_{0\leq t<\infty} \max_{1\leq i,j\leq N} |x_i(t) - x_j(t)| < \infty \quad \text{and} \quad \max_{1\leq i,j\leq N} |v_i(t) - v_j(t)| \leq Ce^{-Ct} \quad \text{for } t \geq 0.$$

Ref.- Cucker-Smale(2007), Ha-Tadmor(2008), Ha-Liu(2009), Carrillo-Fornasier-Rosado-Toscani(2010), ...

Numerical Simulation



by Sergio Pérez

Kinetic Cucker-Smale equation

When $N \to \infty$, the macroscopic observables for Cucker-Smale model can be calculated from the velocity moments of the density function $f = (x, v, t)$ which is a solution to the following Vlasov-type equation:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [F(f)f] = 0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,$$
$$F(f)(x, v, t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x - y)(w - v) f(y, w, t)\, dydw.$$

Flocking:

$$\int |v - v_c|^2 f\, dxdv \to 0 \quad \text{as} \quad t \to \infty \quad \text{where} \quad v_c = \frac{\int v\, f\, dxdv}{\int f\, dxdv}.$$

Several drawbacks:

- Collision between individuals
- Dynamics away from equilibrium
- Vision geometrical constraints
- ....

Ref.- Cucker-Dong(2010), Park-Kim-Ha(2010), Motsch-Tadmor(2011), Ahn-Choi-Ha-Lee(2012), Agueh-Illner-Richardson(2011), ...

Collision avoidance between individuals

Previous work

Cucker-Smale model with collision avoiding forces:

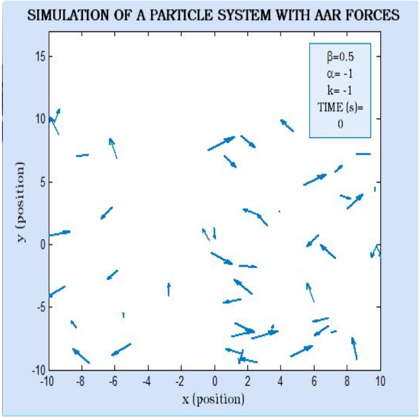
$$\frac{dx_i}{dt} = v_i, \quad t > 0, \ i = 1, \cdots, N,$$

$$\frac{dv_i}{dt} = \underbrace{\frac{1}{N}\sum_{j\neq i}^{N} \psi(x_i - x_j)(v_j - v_i)}_{Alignment} - \underbrace{\frac{1}{N}\sum_{j\neq i}^{N} \nabla K(x_i - x_j)}_{Attraction/Repulsion}, \quad \psi(x) = \frac{1}{(1+|x|^2)^{\beta/2}}.$$

Ref.- Cucker-Dong(2010), Ha-Park-Kim(2010), Yang-Chen(2014), ...

Numerical simulation

- $K(x) = k\phi(x) - \alpha\frac{|x|^2}{2}$ where $\Delta\phi = \delta_0$.



Numerical simulation

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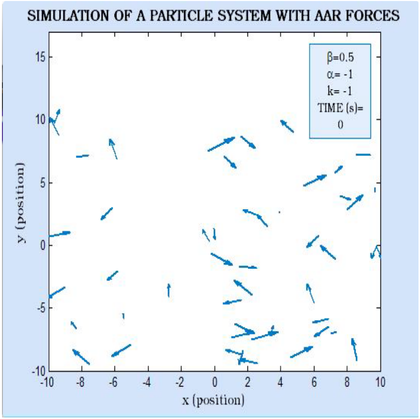

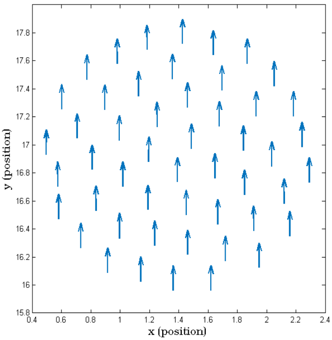

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Singular communication weights

Cucker-Smale model with singular communication weights:

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \;\; i = 1, \cdots, N,$$
$$\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} \psi(x_j - x_i)(v_j - v_i), \quad \psi(x) = \psi^s(x) := \frac{1}{|x|^\alpha}, \quad \alpha > 0. \tag{1}$$

Singular communication weights

Cucker-Smale model with singular communication weights:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad t > 0, \; i = 1, \cdots, N, \\ \frac{dv_i}{dt} &= \frac{1}{N}\sum_{j=1}^{N} \psi(x_j - x_i)(v_j - v_i), \quad \psi(x) = \psi^s(x) := \frac{1}{|x|^\alpha}, \quad \alpha > 0. \end{aligned} \tag{1}$$

Flocking estimate: Let $\alpha \geq 1$. Suppose that the initial data satisfies

$$\|v_0 - v_c(0)\|_{\ell^\infty} < \frac{1}{2}\int_{2\|x_0 - x_c(0)\|_{\ell^\infty}}^{\infty} \psi(s)\, ds.$$

Then there exist positive constants $c_1 > c_0 > 0$ such that

$$\|x(t) - x_c(0)\|_{\ell^\infty} \in [c_0, c_1] \quad \text{and} \quad \|v(t) - v_c(0)\|_{\ell^\infty} \leq \|v_0 - v_c(0)\|_{\ell^\infty}\, e^{-\psi(2c_1)t}.$$

Ref.- Ahn-Choi-Ha-Lee(2012)

Sharp condition to avoid collision

Finite-time collision estimate($\alpha < 1$)**:** Let $d = 1$. If we consider the two-particle system, then the following statements are equivalent:

1. There exists a time $t_0 < \infty$ such that $\phi(t_0) = \dot{\phi}(t_0) = 0$.
2. Initial data satisfy
$$\dot{\phi}(0) = -2\Psi(\phi(0)),$$
where $\Psi(s) := \frac{1}{1-\alpha} s^{1-\alpha}$ is a primitive of ϕ and $\phi(t) = x_1(t) - x_2(t)$.

Non collision estimate($\alpha \geq 1$)**:** Suppose that the initial data (x_0, v_0) are non-collisional, i.e., they satisfy

$$x_{i0} \neq x_{j0} \quad \text{for } 1 \leq i \neq j \leq N.$$

Then the system (1) admits a unique smooth solution. Moreover, the trajectories of this solution are also non-collisional, i.e.,

$$x_i(t) \neq x_j(t) \quad \text{for } 1 \leq i \neq j \leq N, \quad t \geq 0.$$

Ref.- Peszek(2014), Carrillo-C.-Mucha-Peszek(work in progress)

Cucker-Smale model with singular communication weights

Main equation:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \big(F(f)f\big) = 0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \tag{2}$$

where F denotes the alignment force between particles:

$$F(f)(x, v, t) = \int \psi(x - y)(w - v)f(y, w)\, dydw, \quad \psi(x) = \frac{1}{|x|^\alpha} \text{ with } \alpha \in (0, d-1).$$

Definition: For a given $T \in (0, \infty)$, f is a weak solution of (3) on the time-interval $[0, T)$ if and only if the following condition are satisfied:

(i) $f \in L^\infty(0, T; (L^1_+ \cap L^p)(\mathbb{R}^d \times \mathbb{R}^d)) \cap \mathcal{C}([0, T]; \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$,

(ii) f satisfies the equation (3) in the weak sense.

Local existence and uniqueness of weak solutions

Theorem(Carrillo-C.-Hauray, 2014): Suppose that f^0 is compactly supported in velocity and $(\alpha+1)p' < d$, and the initial data f^0 satisfies

$$f^0 \in (L^1_+ \cap L^p)(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d).$$

Then there exist $T > 0$ and unique weak solution f to the system (3) in the sense of Definition 1 on the time interval $[0, T]$. Furthermore, if $f_i, i = 1, 2$ are two such solutions to (3), then we have the following d_1-stability estimate.

$$\frac{d}{dt} d_1(f_1(t), f_2(t)) \leq C d_1(f_1(t), f_2(t)), \quad \text{for} \quad t \in [0, T],$$

where C is a positive constant.

Idea of Proof

(Regularization) The approximate solutions are obtained by solving:

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \nabla_v \cdot \left[F^\varepsilon(f_\varepsilon) f_\varepsilon\right] = 0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \\ F^\varepsilon(f_\varepsilon)(x, v, t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_\varepsilon(x-y)(w-v) f_\varepsilon(y, w, t)\, dy dw, \\ f_\varepsilon(x, v, 0) =: f^0(x, v). \end{cases}$$

where $\psi_\varepsilon := \psi \star \theta_\varepsilon$.

(Support estimate of f_ε in velocity) Consider the forward bi-characteristics $(X_\varepsilon(s; 0, x, v), V_\varepsilon(s; 0, x, v))$ satisfying the following ODE system:

$$\frac{dX_\varepsilon(s)}{ds} = V_\varepsilon(s),$$
$$\frac{dV_\varepsilon(s)}{ds} = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_\varepsilon\left(X_\varepsilon(s) - y\right)\left(w - V_\varepsilon(s)\right) f_\varepsilon(y, w, s)\, dy dw.$$

Set $\Omega_\varepsilon(t)$ and $R^v_\varepsilon(t)$ the v-projection of compact supp$f_\varepsilon(\cdot, t)$ and maximum value of v in $\Omega_\varepsilon(t)$, respectively:

$$\Omega_\varepsilon(t) := \overline{\{v \in \mathbb{R}^d : \exists (x, v) \in \mathbb{R}^d \times \mathbb{R}^d \text{ such that } f_\varepsilon(x, v, t) \neq 0\}}, \quad R^v_\varepsilon(t) := \max_{v \in \Omega_\varepsilon(t)} |v|.$$

Then we have

$$R^v_\varepsilon(t) \le R^v_\varepsilon(0) = R^v_0 := \max_{v \in \Omega(0)} |v|.$$

Idea of Proof(Conti.)

(Uniform bound estimate): Let f_ε be the solution to regularized equation. Then there exists a $T > 0$ such that the uniform $L^1 \cap L^p$-estimate of f_ε

$$\sup_{t \in [0,T]} \|f_\varepsilon\|_{L^1 \cap L^p} \leq C,$$

holds, where C is a positive constant independent of ε.

(f_ε is a Cauchy sequence): Let f_ε and $f_{\varepsilon'}$ be two solutions of the regularized equation. Then there exists C independent of ε and ε' such that

$$\frac{d}{dt} d_1(f_\varepsilon(t), f_{\varepsilon'}(t)) \leq C(d_1(f_\varepsilon(t), f_{\varepsilon'}(t)) + \varepsilon + \varepsilon'),$$

holds for all $t \geq 0$.

(Passing to the limit $\varepsilon \to 0$): The limit curve of measure f obtained from the above is a solution of the equation (3).

Open question: Rigorous derivation of the mean-field limit

Further extensions: Flocking models with singular kernels

Main equation:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \big(F(f)f\big) = 0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \tag{3}$$

where F denotes the alignment force between particles:

$$F(f)(x, v, t) = -\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x - y)\nabla_v \phi(v - w) f(y, w, t)\, dy dw.$$

Here, the potential function $\phi(v)$ for the velocity coupling is given by

$$\phi(v) = \frac{1}{\beta}|v|^\beta, \quad \text{where} \quad \beta > 0.$$

Two different cases:

- Singular communication weight and super-linear velocity coupling:

$$\psi(x) = \psi^1(x) := \frac{1}{|x|^\alpha} \quad \text{with} \quad \alpha \in (0, d-1) \quad \text{and} \quad \beta \geq 2.$$

- Regular communication weight and sub-linear velocity coupling:

$$\psi(x) = \psi^2(x) \geq 0 \quad \text{symmetric, with} \quad \psi^2 \in \left(L^\infty_{loc} \cap \mathrm{Lip}_{loc}\right)(\mathbb{R}^d) \quad \text{and} \quad \beta \in \left(\frac{3-d}{2}, 2\right).$$

Normalized communication weights

Cucker-Smale model:

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \ i = 1, \cdots, N,$$
$$\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} \psi(x_j - x_i)(v_j - v_i), \quad \psi(x) = \frac{1}{(1 + |x|^2)^{\beta/2}}.$$

Cucker-Smale model with normalized communication weights:

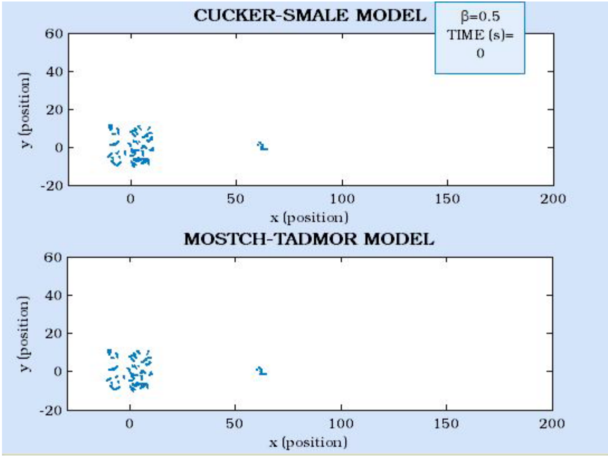
$$\frac{dx_i}{dt} = v_i, \quad t > 0, \ i = 1, \cdots, N,$$
$$\frac{dv_i}{dt} = \frac{1}{\sum_{k=1}^{N} \psi(x_i - x_k)} \sum_{j=1}^{N} \psi(x_j - x_i)(v_j - v_i).$$

- Motsch-Tadmor(2011)

$$\int^{\infty} \psi^2(r)\, dr = \infty \quad \Longrightarrow \quad \text{Flocking}$$

Ref.- Tadmor-Tan (2014): $\int^{\infty} \psi(r)\, dr = \infty \quad \Longrightarrow \quad$ Flocking

Numerical simulation



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Interacting diffusing particle systems and kinetic equation

Interacting diffusing particle system:

$$dx_i = v_i dt, \quad t > 0, \ i = 1, \cdots, N,$$

$$dv_i = \frac{1}{\sum_{k=1}^{N} \psi(x_i - x_k)} \sum_{j=1}^{N} \psi(x_j - x_i)(v_j - v_i)dt + \sqrt{2} dB_i.$$

Interacting diffusing particle systems and kinetic equation

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Kinetic model:

$$\partial_t F + v \cdot \nabla_x F + \nabla_v \cdot ((\tilde{u}_F - v)F) = \Delta_v F, \quad x \in \mathbb{R}^d, \ \ v \in \mathbb{R}^d,$$

where $\tilde{u}_F$ is given by

$$\tilde{u}_F(x,t) := \frac{(\psi \star b^F)(x,t)}{(\psi \star a^F)(x,t)},$$

with

$$a^F(x,t) := \int_{\mathbb{R}^d} F(x,v,t)\,dv \quad \text{and} \quad b^F(x,t) := \int_{\mathbb{R}^d} vF(x,v,t)\,dv.$$

Interacting diffusing particle systems and kinetic equation

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with

$$a^F(x,t) := \int_{\mathbb{R}^d} F(x,v,t)\, dv \quad \text{and} \quad b^F(x,t) := \int_{\mathbb{R}^d} vF(x,v,t)\, dv.$$

Consider the singular limit in which the communication weight ψ converges to a Dirac distribution.

$$\tilde{u}_F \to u_F := \frac{b^F}{a^F} = \frac{\int_{\mathbb{R}^d} vF\, dv}{\int_{\mathbb{R}^d} F\, dv}.$$

Ref.- Karper-Mellet-Trivisa(2014)

Perturbation framework

Nonlinear Fokker-Planck equation:

$$\partial_t F + v \cdot \nabla_x F = \nabla_v \cdot (\nabla_v F + (v - u_F)F).$$

Define the perturbation $f = f(x, v, t)$ by

$$F = M + \sqrt{M} f, \quad M = M(v) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|v|^2}{2}\right)$$

The equation for the perturbation f satisfies

$$\partial_t f + v \cdot \nabla_x f + u_F \cdot \nabla_v f = \mathbf{L} f + \Gamma(f, f), \tag{4}$$

where the linear part $\mathbf{L}f$ and the nonlinear part $\Gamma(f, f)$ are given by

$$\mathbf{L} f := \frac{1}{\sqrt{M}} \nabla_v \cdot \left(M \nabla_v \left(\frac{f}{\sqrt{M}} \right) \right) \quad \text{and} \quad \Gamma(f, f) = u_F \cdot \left(\frac{1}{2} v f + v\sqrt{M} \right),$$

respectively.

$\Gamma(f, f)$ is not bilinear!

Ref.- Duan-Fornasier-Toscani(2010), Karper-Mellet-Trivisa(2013)

Global classical solutions and time-asymptotic behavior

Theorem(C. 2015): Let $d \geq 3$ and $s \geq 2[d/2] + 2$. Suppose $F_0 \equiv M + \sqrt{M} f_0 \geq 0$ and $\|f_0\|_{H^s} \leq \epsilon_0 \ll 1$. Then we have the global existence of the unique classical solution $f(x, v, t)$ to the equation (4) satisfying

$$f \in \mathcal{C}([0, \infty); H^s(\mathbb{R}^d \times \mathbb{R}^d)), \quad F \equiv M + \sqrt{M} f \geq 0,$$

and

$$\|f(t)\|_{H^s}^2 + c_1 \int_0^t \|\nabla_x (a, b)\|_{H^{s-1}}^2 ds + c_1 \int_0^t \sum_{0 \leq k+l \leq s} \|\nabla_x^k \nabla_v^l \{\mathbf{I} - \mathbf{P}\} f\|_{\mu}^2 ds \leq c_2 \|f_0\|_{H^s}^2,$$

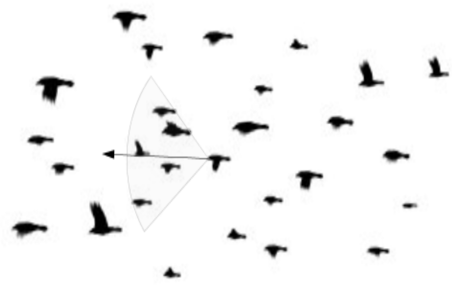
for some positive constants $c_1, c_2 > 0$. Furthermore, if $\|f_0\|_{L_v^2(L^1)}$ is bounded, we have

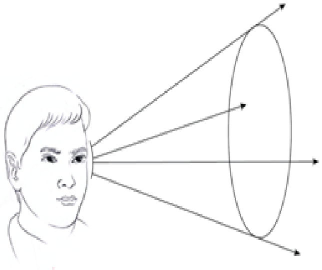
$$\|f(t)\|_{H^s} \leq C \left(\|f_0\|_{H^s} + \|f_0\|_{L_v^2(L^1)} \right) (1 + t)^{-\frac{d}{4}}, \quad t \geq 0,$$

where C is a positive constant independent of t.

Idea of proof: Careful analysis of the nonlinear dissipation term + Coercivity of the linear operator $\mathbf{L}$ + Hypocoercivity for a modified linearized Cauchy problem with a non-homogeneous microscopic source.

Collective behavior models with sharp sensitivity regions





Collective behavior models at particle and kinetic levels

Particle system:

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \ i = 1, \cdots, N,$$
$$\frac{dv_i}{dt} = \sum_{j \neq i} m_j \mathbf{1}_{K(v_i)}(x_i - x_j)(v_j - v_i),$$

where $\mathbf{1}_{K(v_i)}$ is the indicator function on the set $K(v_i)$. Here $\{x_i\}_{i=1}^N$, $\{v_i\}_{i=1}^N$, $\{m_i\}_{i=1}^N$ are the position, velocity, and weight of i-th particle, respectively.

Kinetic equation: As a mean-field limit, we consider

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [F(f)f] = 0, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \ t > 0, \tag{5}$$

where $F(f)$ is a alignment force of velocities given by

$$F(f)(x, v, t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{K(v)}(x - y)(w - v) f(y, w)\, dydw.$$

Main assumptions

(**H1**): $K(v)$ is globally compact, i.e., $K(v)$ is compact and there exists a compact set K such that $K(v) \subseteq K$ for all $v \in \mathbb{R}^d$.

(**H2**): There exists a family of closed sets $v \mapsto \Theta(v)$ and a constant C such that:

- $\partial K(v) \subset \Theta(v)$, for all $v \in \mathbb{R}^d$,
- $|\Theta(v)^{\varepsilon,+}| \leq C\varepsilon$, for all $\varepsilon \in (0,1)$,
- $K(v) \Delta K(w) \subset \Theta(v)^{C|v-w|,+}$, for $v, w \in \mathbb{R}^d$,
- $\Theta(w) \subset \Theta(v)^{C|v-w|,+}$, for $v, w \in \mathbb{R}^d$.

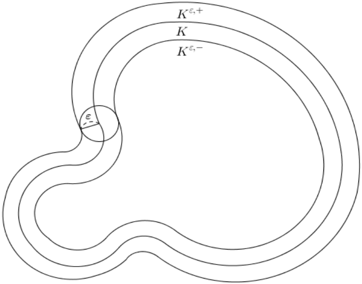
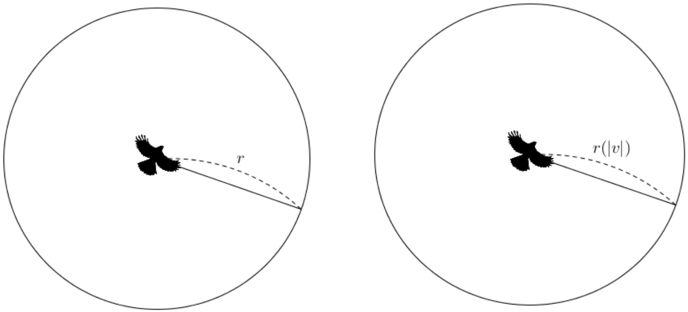


Figure: A sketch of the sets K, $K^{\varepsilon,+}$, and $K^{\varepsilon,-}$

Examples



- A ball with a fixed radius:

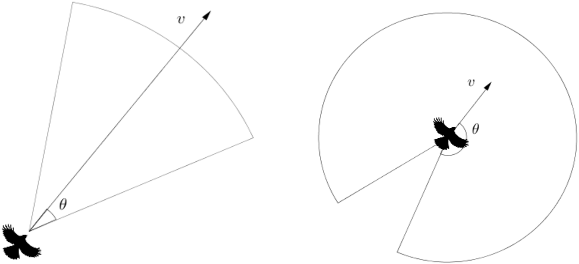
$$K(v) \equiv B(0,r) \quad \text{where} \quad B(0,r) := \{x \in \mathbb{R}^d : |x| \leq r\} \quad \text{with} \quad r > 0.$$

- A ball with a radius depending on the speed:

$$K(v) = B(0, r(|v|)) \quad \text{where} \quad r : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is bounded and Lipschitz.}$$

$$\Theta(v) = \partial K(v)$$

Examples



- A vision cone:

$$K(v) = C(r, v, \theta(|v|)) := \left\{ x : |x| \leq r \quad \text{and} \quad -\theta(|v|) \leq \cos^{-1}\left(\frac{x \cdot v}{|x||v|}\right) \leq \theta(|v|) \right\},$$

with $0 < \theta(z) \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $\theta(z) = \pi$ for $0 \leq z \leq 1$, $\theta(z)$ is decreasing for $z \geq 1$, and $\theta(z) \to \theta_* > 0$ as $|z| \to +\infty$.

$$\Theta(v) := \begin{cases} \partial C(r, v, \theta(|v|)) \cup R(v) & \text{if } |v| \in (1/2, 1), \\ \partial C(r, v, \theta(|v|)) & \text{else ,} \end{cases}$$

where $R(v) = [a(v), b(v)]$ with

$$a(v) = -r\frac{v}{|v|} \quad , \quad b(v) = 2r(|v| - 1)\frac{v}{|v|}.$$

Global weak solutions & stability

Definition: For a given $T \in (0, \infty)$, f is a weak solution of (5) on the time-interval $[0, T)$ if and only if the following condition are satisfied:

$$(i) \quad f \in L^\infty(0, T; (L^1_+ \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d)) \cap \mathcal{C}([0, T]; \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)),$$

(ii) f satisfies the equation (5) in the weak sense.

Theorem(Carrillo-C.-Hauray-Salem 2016): Given an initial data satisfying

$$f_0 \in (L^1_+ \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)\,, \tag{6}$$

and assume further that f_0 is compactly supported in velocity. Then there exists a positive time $T > 0$ such that the system (5) with the sensitivity region set-valued function $K(v)$ satisfying (**H1**)-(**H2**) admits a unique weak solution $f \in L^\infty(0, T : (L^1_+ \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d))$, which is also compactly supported in velocity. Furthermore if $f_i, i = 1, 2$ are two such solutions to the system (5) with initial data $f_i(0)$ satisfying (6), we have

$$d_1(f_1(t), f_2(t)) \le d_1(f_1(0), f_2(0))e^{Ct}, \quad \text{for} \quad t \in [0, T],$$

where C is a positive constant that depends only on the $L^\infty(\mathbb{R}^d \times \mathbb{R}^d \times (0, T))$ norm of f_1.

Weak-strong stability!

Further extensions

Second-order model:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F(f)f) = 0, \qquad (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0,$$

where the force term $F(f)$ can be chosen from two different types:

$$F(f)(x,v,t) := \begin{cases} \displaystyle\int \psi(x-y)\mathbf{1}_{K(v)}(x-y)h(w-v)f(y,w)\,dydw & \text{(Cucker-Smale)} \\ \displaystyle\int \nabla_x \varphi(x-y)\mathbf{1}_{K(v)}(x-y)f(y,w)\,dydw & \text{(Attractive-Repulsive).} \end{cases}$$

Here ψ, h, and φ denote the communication weight, velocity coupling, and interaction potential, respectively.

First-order model:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \qquad x \in \mathbb{R}^d, \quad t > 0,$$

where the vector field u is given by

$$u(x,t) := \int_{\mathbb{R}^d} \mathbf{1}_{K(w(x))}(x-y)\nabla_x \varphi(x-y)\rho(y)\,dy,$$

and $\varphi \in W^{1,\infty}(\mathbb{R}^d)$ and w is a given orientational field satisfying $w \in W^{1,\infty}(\mathbb{R}^d)$ and $|w| \geq w_0 > 0$.

Thank you for your attention