Convergence to equilibrium for linear and homogeneous Fokker-Planck equations

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Fokker-Planck equations

$$\partial_t f = \mathcal{D}_\varepsilon f + \mathrm{div}(vf) =: \Lambda_\varepsilon f, \quad \varepsilon \geq 0 \qquad (\mathsf{FP}_\varepsilon)$$

- $f = f(t,v) \geq 0$ is the homogeneous density of particles with $t \in \mathbb{R}^+$ the time and $v \in \mathbb{R}^d$ the velocity.
- $\mathcal{D}_\varepsilon$ is a diffusion operator which can be

Discrete	$\mathcal{D}_\varepsilon(f) := (k_\varepsilon * f - f)/\varepsilon^2, \quad \varepsilon > 0$
Fractional	$\mathcal{D}_\varepsilon(f) := -(-\Delta)^{(2-\varepsilon)/2} f, \quad \varepsilon > 0$
Classical	$\mathcal{D}_0(f) := \Delta f$

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We have $\mathcal{D}_\varepsilon \to \mathcal{D}_0$.

Properties of the FP equations

Consider f_t a solution of $(\mathrm{FP}_\varepsilon)$ with initial datum $f_0 = f$.

- Conservation of mass:

$$\langle f_t \rangle = \langle f \rangle, \quad \forall\, t \geq 0 \quad \text{where} \quad \langle g \rangle := \int_{\mathbb{R}^d} g\, dv.$$

- Conservation of positivity:

$$f \geq 0 \Rightarrow f_t \geq 0, \quad \forall\, t \geq 0.$$

- Unique positive stationary state with unit mass:

$$\exists!\, G_\varepsilon, \quad \Lambda_\varepsilon G_\varepsilon = 0 \quad \text{and} \quad \langle G_\varepsilon \rangle = 1.$$

- Stationary state exponentially stable:

$$f_t - \langle f \rangle G_\varepsilon \to 0 \quad \text{as} \quad t \to \infty \quad \text{with an exponential rate.}$$

Spectral analysis of self-adjoint operators, Poincaré or logarithmic Sobolev inequalities, Krein-Rutman theory for positive semigroups...

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$\hookrightarrow$ Can we handle the problem of convergence to equilibrium for these equations uniformly with respect to ε?

Main result

Theorem (Mischler-T. '15)

There exists $\varepsilon_0 \in (0,2)$, $a < 0$ *and* $C \geq 1$ *such that for any* $f \in X$ *and any* $\varepsilon \in [0, \varepsilon_0]$,

$$\|S_{\Lambda_\varepsilon}(t) f - \langle f \rangle G_\varepsilon\|_X \leq C\, e^{at} \|f - \langle f \rangle G_\varepsilon\|_X, \quad \forall\, t \geq 0$$

where

- $S_{\Lambda_\varepsilon}(t)$ *is the semigroup associated to the generator* Λ_ε,
- X *is a weighted* L^1 *space independent of* ε,
- G_ε *is the equilibrium of the equation of mass* 1.

Ideas of the proof

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(1) Argument of enlargement of the space where the linear operator has a spectral gap: we know that Λ has a spectral gap in E, we prove that Λ also has a spectral gap in $\mathcal{E} \supset \mathsf{E}$. [Mouhot '06] [Gualdani-Mischler-Mouhot '13]

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(1) Argument of enlargement of the space where the linear operator has a spectral gap: we know that Λ has a spectral gap in E, we prove that Λ also has a spectral gap in $\mathcal{E} \supset E$.
[Mouhot '06] [Gualdani-Mischler-Mouhot '13]

(2) Perturbative argument for $\Lambda = \Lambda_\varepsilon$ such that $\Lambda_\varepsilon \to \Lambda_0$: we know that Λ_0 has a spectral gap, we prove that for ε small enough , Λ_ε also has a spectral gap in the same space.
[Mischler-Mouhot '09] [T. '16]

Ideas of the proof

For both arguments, we have to exhibit a splitting $\Lambda = \mathcal{A} + \mathcal{B}$ which satisfies:

- $\mathcal{A}$ regular
- $\mathcal{B}$ (hypo)dissipative
- $\mathcal{A} S_{\mathcal{B}}(t)$ regularizing.

The equations

Discrete Fokker-Planck equation

$$\partial_t f = \frac{1}{\varepsilon^2}(k_\varepsilon * f - f) + \mathrm{div}(vf) =: \Lambda_\varepsilon f, \quad \varepsilon > 0$$

with $k \in W^{2,1}(\mathbb{R}^d) \cap L^1(\langle v \rangle^{r_0})$ with $r_0 > 0$ large enough which satisfies

$$\int_{\mathbb{R}^d} k(v) \begin{pmatrix} 1 \\ v \\ v \otimes v \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ 2I_d \end{pmatrix} \quad \text{and} \quad k_\varepsilon(v) := \frac{1}{\varepsilon^d} k\left(\frac{v}{\varepsilon}\right),\ v \in \mathbb{R}^d.$$

Classical Fokker-Planck equation

$$\partial_t f = \Delta f + \mathrm{div}(vf) =: \Lambda_0 f$$

Main result

Theorem (Mischler-T. '15)

Consider $q > d/2$. There exists $\varepsilon_0 > 0$, $a_0 < 0$ such that: $\forall\, \varepsilon \in [0, \varepsilon_0]$, $\forall\, f \in L^1(\langle v \rangle^q)$, $\forall\, a > a_0$,

$$\|S_{\Lambda_\varepsilon}(t) f - \langle f \rangle\, G_\varepsilon\|_{L^1(\langle v \rangle^q)} \le C_a\, e^{at} \|f - \langle f \rangle\, G_\varepsilon\|_{L^1(\langle v \rangle^q)}, \quad \forall\, t \ge 0$$

where $S_{\Lambda_\varepsilon}(t)$ is the semigroup associated to the generator Λ_ε, G_ε is the equilibrium of the equation of mass 1.

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where $S_{\Lambda_\varepsilon}(t)$ is the semigroup associated to the generator Λ_ε, G_ε is the equilibrium of the equation of mass 1.

- First step of the proof: use a perturbative argument to obtain a result in $H^3(\langle v \rangle^r)$.
- Second step of the proof: use an enlargement argument to conclude.

Step 1 of the proof: perturbative argument

- Spaces at stake: $m(v) = \langle v \rangle^r, \quad r > d/2 + 6$

$$X_1 := H^6(m) \subset X_0 := H^3(m) \subset X_{-1} := L^2(m).$$

Step 1 of the proof: perturbative argument

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- Consider $\chi_R \in \mathcal{D}(\mathbb{R}^d)$, $\mathbb{1}_{B(0,R)} \leq \chi_R \leq \mathbb{1}_{B(0,2R)}$ and $\chi_R^c := 1 - \chi_R$. For $\varepsilon > 0$, we define

$$\mathcal{A}_\varepsilon f := M \chi_R (k_\varepsilon * f) \quad \text{and} \quad \mathcal{B}_\varepsilon f := \Lambda_\varepsilon f - \mathcal{A}_\varepsilon f.$$

For $\varepsilon = 0$, we define

$$\mathcal{A}_0 f := M \chi_R f \quad \text{and} \quad \mathcal{B}_0 f := \Lambda_0 f - \mathcal{A}_0 f.$$

Convergence of the splitting

We prove that:

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{\mathcal{B}(X_i, X_{i-1})} \xrightarrow[\varepsilon\to 0]{} 0 \quad \text{and} \quad \|\Lambda_\varepsilon - \Lambda_0\|_{\mathcal{B}(X_i, X_{i-1})} \xrightarrow[\varepsilon\to 0]{} 0.$$

We have:

$$\mathcal{A}_\varepsilon f - \mathcal{A}_0 f = M\chi_R(k_\varepsilon * f - f) \quad \text{and} \quad \Lambda_\varepsilon - \Lambda_0 = \mathcal{D}_\varepsilon - \Delta$$

Taylor expansion to deal with $\mathcal{D}_\varepsilon - \Delta$.

Properties of the limit operator Λ_0

Theorem (Gualdani-Mischler-Mouhot '13 and Mischler-Mouhot '14)

Consider $i \in \{-1, 0, 1\}$. There exists $a_0 < 0$ such that for any $f \in X_i$ and any $a > a_0$,

$$\|S_{\Lambda_0}(t)f - G_0\langle f\rangle\|_{X_i} \leq C_a\, e^{at}\, \|f - G_0\langle f\rangle\|_{X_i}, \quad \forall\, t \geq 0$$

where $S_{\Lambda_0}(t)$ is the semigroup associated to the generator Λ_0 and G_0 is the unique equilibrium of the equation of mass 1.

Regularity of $\mathcal{A}_\varepsilon$ and dissipativity of $\mathcal{B}_\varepsilon$ - I

- $\mathcal{A}_\varepsilon$ is bounded in X_i, $i = -1, 0, 1$ uniformly with respect to ε.
- Let $a > d/2 - q + 6$. There exists $\varepsilon_0 > 0$, $M \geq 0$ and $R \geq 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, $\mathcal{B}_\varepsilon - a$ is hypodissipative in X_i, $i = -1, 0, 1$ with a which does not depend on ε.

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The example of $L^2(m)$. Consider $f_t := S_{\mathcal{B}_\varepsilon}(t) f$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|f_t\|^2_{L^2(m)} = \int_{\mathbb{R}^d} \Big(\mathcal{B}_\varepsilon f_t\Big) f_t \, m^2.$$

$\hookrightarrow$ We are thus going to estimate $\int_{\mathbb{R}^d} \Big(\mathcal{B}_\varepsilon f\Big) f \, m^2$.

Dissipativity of $\mathcal{B}_\varepsilon$ - II

We have:

$$\mathcal{B}_\varepsilon f = \frac{1}{\varepsilon^2}(k_\varepsilon * f - f) + \mathrm{div}(vf) - M\chi_R(k_\varepsilon * f)$$
$$= \left(\frac{1}{\varepsilon^2} - M\right)(k_\varepsilon * f - f) + M\chi_R^c(k_\varepsilon * f - f) + \mathrm{div}(vf) - M\chi_R f.$$

Estimate of $\int_{\mathbb{R}^d}(\mathcal{B}_\varepsilon f) f\, m^2$:

$$\begin{aligned}\int_{\mathbb{R}^d}(\mathcal{B}_\varepsilon f) f\, m^2 &= \left(\frac{1}{\varepsilon^2} - M\right)\int_{\mathbb{R}^d}(k_\varepsilon * f - f) f\, m^2 \\ &\quad + \int_{\mathbb{R}^d} M\chi_R^c(k_\varepsilon * f - f) f\, m^2 \\ &\quad + \int_{\mathbb{R}^d}\mathrm{div}(vf) f\, m^2 - \int_{\mathbb{R}^d} M\chi_R f^2\, m^2 \\ &=: T_1 + T_2 + T_3 + T_4.\end{aligned}$$

Dissipativity of $\mathcal{B}_\varepsilon$ - III

We have the following estimates:

$$T_1 \le \text{non positive term} + C \int_{\mathbb{R}^d} f^2(v)\, m^2(v)\, \frac{1}{\langle v \rangle^2}\, dv$$

$$T_2 \le M\, C_R\, \kappa_\varepsilon \int_{\mathbb{R}^d} f^2\, m^2, \qquad \kappa_\varepsilon \xrightarrow[\varepsilon \to 0]{} 0$$

$$T_3 + T_4 = \int_{\mathbb{R}^d} f^2(v)\, m^2(v) \left(\frac{d}{2} - \frac{r\,|v|^2}{\langle v \rangle^2} - M \chi_R(v) \right) dv$$

Dissipativity of $\mathcal{B}_\varepsilon$ - III

We obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f) f\, m^2 \\
&\leq \int_{\mathbb{R}^d} f^2\, m^2 \Bigg(\underbrace{C \langle v \rangle^{-2} + \frac{d}{2} - \frac{r\,|v|^2}{\langle v \rangle^2} + M\, C_R\, \kappa_\varepsilon}_{\xrightarrow[\varepsilon \to 0, |v| \to \infty]{} d/2 - r < 0} - M \chi_R \Bigg)
\end{aligned}$$

Dissipativity of $\mathcal{B}_\varepsilon$ - III

We obtain:

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f) f\, m^2$$
$$\leq \int_{\mathbb{R}^d} f^2 m^2 \Bigg(\underbrace{C\langle v\rangle^{-2} + \frac{d}{2} - \frac{r|v|^2}{\langle v\rangle^2} + M C_R \kappa_\varepsilon}_{\xrightarrow[\varepsilon\to 0, |v|\to\infty]{} d/2 - r < 0} - M\chi_R \Bigg)$$

For $a > d/2 - r$, one can find M, R, ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon - a) f\, f\, m^2 \leq 0$$

and thus

$$\left\| S_{\mathcal{B}_\varepsilon}(t) f \right\|_{L^2(m)} \leq e^{at} \|f\|_{L^2(m)}.$$

Regularization properties of $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)$

We prove that there exists $n \in \mathbb{N}$ such that:

$$\|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{X_i \to X_{i+1}} \leq C_a\, e^{at}, \quad i = -1, 0.$$

We consider $f_t = S_{\mathcal{B}_\varepsilon}(t) f$. We have, if $\varepsilon_0 > 0$ is small enough, for any $\varepsilon \in (0, \varepsilon_0]$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f_t\|^2_{L^2(m)} &= \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f_t) f_t\, m^2 \\ &\leq \underbrace{-\frac{1}{4\varepsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (f_t(y) - f_t(v))^2\, k_\varepsilon(v-y)\, dy\, dv}_{\text{gain of regularity}} + a\|f_t\|^2_{L^2(m)}\end{aligned}$$

Summary

There exist $a_0 < 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$:

- For any $i = -1, 0, 1$, $\mathcal{A}_\varepsilon \in \mathscr{B}(X_i)$ uniformly in ε.
- For any $a > a_0$, there exists $C_a > 0$ such that

$$\forall\, i = -1, 0, 1, \quad \forall\, t \geq 0, \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{X_i \to X_i} \leq C_a\, e^{at}.$$

- For any $a > a_0$, there exist $n \geq 1$ and $C_{n,a} > 0$ such that

$$\forall\, i = -1, 0, \quad \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{X_i \to X_{i+1}} \leq C_{n,a}\, e^{at}.$$

- There exists a function $\eta(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0$ such that

$$\forall\, i = -1, 0, \quad \|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{X_i \to X_i} + \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{X_i \to X_{i-1}} \leq \eta(\varepsilon).$$

- $\Sigma(\Lambda_0) \cap \{z \in \mathbb{C},\, \mathfrak{Re}\, z > a_0\} = \{0\}$ in spaces X_i, $i = -1, 0, 1$, where 0 is a one dimensional eigenvalue.

Conclusion - I

Thanks to the perturbative argument [Mischler-Mouhot '09] and [T. '16], we obtain the existence of $a_0 < 0$ and $\varepsilon_0 > 0$ such that:

for any $\varepsilon \in [0, \varepsilon_0]$, for any $a > a_0$ and for any $f \in X_0 = H^3(\langle v \rangle^r)$,

$$\|S_{\Lambda_\varepsilon}(t)f - \langle f \rangle \, G_\varepsilon\|_{X_0} \leq C_a \, e^{at} \|f - \langle f \rangle \, G_\varepsilon\|_{X_0}, \quad \forall \, t \geq 0.$$

Step 2 of the proof: enlargement argument

$$\text{Small space:} \quad \mathrm{E} := H^3(\langle v\rangle^r)$$
$$\text{Large space:} \quad \mathcal{E} := L^1(\langle v\rangle^q)$$

Using the same splitting as previously, we have:

- ⋆ $\mathcal{A} \in \mathscr{B}(\mathrm{E})$ and $\mathcal{A} \in \mathscr{B}(\mathcal{E})$.
- ⋆ $\mathcal{B} - a$ is hypodissipative in $\mathcal{E}$.
- ⋆ There exist $n \geq 1$ and $C_{n,a} > 0$ such that

$$\|(\mathcal{A} S_{\mathcal{B}})^{(*n)}(t)\|_{\mathcal{E}\to E} \leq C_{n,a}\, e^{at}.$$

We then use the enlargement argument to conclude.

Conclusion - II

Theorem (Mischler-T. '15)

Consider $q > d/2$. There exists $\varepsilon_0 \in (0,2)$ and $a_0 < 0$ such that $\forall\, \varepsilon \in [0, \varepsilon_0], , \forall\, t \geq 0, \forall\, f \in L^1(\langle v \rangle^q), \forall\, a > a_0,$

$$\|S_{\Lambda_\varepsilon}(t) f - \langle f \rangle\, G_\varepsilon\|_{L^1(\langle v \rangle^q)} \leq C_a\, e^{at} \|f - \langle f \rangle\, G_\varepsilon\|_{L^1(\langle v \rangle^q)}, \quad \forall\, t \geq 0$$

where $S_{\Lambda_\varepsilon}(t)$ is the semigroup associated to the generator Λ_ε, G_ε is the equilibrium of the equation of mass 1.

1) Introduction

2) Discrete and classical Fokker-Planck equations

3) Fractional and classical FP equations

The equations

Main result and idea of the proof

The equations

Fractional Fokker-Planck equation

$$\partial_t f = -(-\Delta)^{\frac{2-\varepsilon}{2}} f + \operatorname{div}(vf) =: \Lambda_\varepsilon f, \qquad \varepsilon \in (0,2)$$

with

$$-(-\Delta)^{\frac{2-\varepsilon}{2}} f(v) := c_\varepsilon \int_{\mathbb{R}^d} \frac{f(y) - f(v) - \chi(y-v)(y-v) \cdot \nabla f(v)}{|y-v|^{d+2-\varepsilon}} \, dy$$

and

$$\frac{c_\varepsilon}{2} \int_{|z| \le 1} \frac{z_1^2}{|z|^{d+2-\varepsilon}} \, dz = 1, \quad \text{which implies} \quad c_\varepsilon \approx \varepsilon.$$

Classical Fokker-Planck equation

$$\partial_t f = \Delta f + \operatorname{div}(vf) =: \Lambda_0 f$$

Main result

Theorem (T. ’15 and Mischler-T. ’15)

Assume $\varepsilon_0 \in (0,2)$ *and* $q < 2 - \varepsilon_0$. *There exists* $a_0 < 0$ *such that* $\forall\, \varepsilon \in [0, 2-\varepsilon_0]$, $\forall\, f \in L^1(\langle v \rangle^q)$, $\forall\, a > a_0$,

$$\|S_{\Lambda_\varepsilon}(t) f - G_\varepsilon \langle f \rangle\|_{L^1(\langle v \rangle^q)} \leq C_a \, e^{at} \|f - G_\varepsilon \langle f \rangle\|_{L^1(\langle v \rangle^q)}, \quad \forall\, t \geq 0$$

where $S_{\Lambda_\varepsilon}(t)$ *is the semigroup associated to the generator* Λ_ε, G_ε *is the equilibrium of the equation of mass* 1.

We use an enlargement argument for each $\varepsilon \in [0, 2-\varepsilon_0]$.

Enlargement argument

Spaces at stake: $m(v) = \langle v \rangle^q, \quad 0 < q < 2 - \varepsilon_0$

$$\mathsf{E}_\varepsilon := L^2(G_\varepsilon^{-1/2}) \subset \mathcal{E} := L^1(m).$$

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Theorem (Gentil-Imbert '08)

There exists a constant $\lambda > 0$ *such that for any* $\varepsilon \in (0,2)$,

- *in* $\mathrm{E}_\varepsilon = L^2(G_\varepsilon^{-1/2})$, *there holds* $\Sigma(\Lambda_\varepsilon) \cap \mathrm{D}_{-\lambda} = \{0\}$;
- *the following estimate holds:* $\forall f \in E_\varepsilon$, $\forall a > -\lambda$,

$$\|S_{\Lambda_\varepsilon}(t) f - G_\varepsilon \langle f \rangle\|_{E_\varepsilon} \le e^{at} \|f - G_\varepsilon \langle f \rangle\|_{E_\varepsilon}, \quad \forall t \ge 0.$$

$\hookrightarrow$ We have a result in E_ε for each $\varepsilon \in [0, 2 - \varepsilon_0]$
with a rate of decay uniform with respect to ε.

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$\hookrightarrow$ We have a result in E_ε for each $\varepsilon \in [0, 2 - \varepsilon_0]$ with a rate of decay uniform with respect to ε.

We define for $\varepsilon \in [0,2]$:

$$\mathcal{A}_\varepsilon f := M \chi_R f \quad \text{and} \quad \mathcal{B}_\varepsilon f := \Lambda_\varepsilon f - \mathcal{A}_\varepsilon f.$$

Dissipativity and regularization properties

- To get the dissipativity properties of $\mathcal{B}_\varepsilon$, we proceed similarly as in the previous part since we have an estimate of type

$$\int_{\mathbb{R}^d} (\Lambda_\varepsilon f)\,\mathrm{sign} f\, m \leq \text{ non positive term } + \int_{\mathbb{R}^d} |f|\, m\, \psi_{m,\varepsilon}$$

with

$$\psi_{m,\varepsilon} = \frac{-(-\Delta)^{\frac{2-\varepsilon}{2}}(m)}{m} - \frac{v\cdot\nabla m}{m} \underset{\infty}{\longrightarrow} -q < 0$$

uniformly in $\varepsilon \in [0, 2-\varepsilon_0]$.

- The regularization properties of $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)$ comes from:
  - a gain in weight thanks to $\mathcal{A}_\varepsilon$,
  - a gain of integrability $L^1 \to L^2$ from $S_{\mathcal{B}_\varepsilon}(t)$ using the fractional Nash inequality

$$\|f\|_{L^2} \leq C \|f\|_{L^1}^{\alpha/(d+\alpha)} \|f\|_{\dot{H}^{\alpha/2}}^{d/(d+\alpha)}.$$

Conclusion

Theorem (Mischler-T. '15)

Assume $\varepsilon_0 \in (0,2)$ and $q < 2 - \varepsilon_0$. There exists $a_0 < 0$ such that $\forall\, \varepsilon \in [0, 2-\varepsilon_0]$, $\forall\, t \geq 0$, $\forall\, f \in L^1(\langle v \rangle^q)$, $\forall\, a > a_0$,

$$\|S_{\Lambda_\varepsilon}(t) f - G_\varepsilon \langle f \rangle\|_{L^1(\langle v \rangle^q)} \leq C_a\, e^{at} \|f - G_\varepsilon \langle f \rangle\|_{L^1(\langle v \rangle^q)}$$

where $S_{\Lambda_\varepsilon}(t)$ is the semigroup associated to the generator Λ_ε, G_ε is the equilibrium of the equation of mass 1.

Thanks for your attention!