On growth fragmentation equations

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The equation

$$\frac{\partial u}{\partial t}(t,x) + \frac{\partial}{\partial x}\big(\tau(x)u(t,x)\big) + B(x)u(t,x) = \int_x^\infty k(y,x)B(y)u(t,y)dy,$$

$u(t,x)$: concentration of particles of size x at time t

$\tau(x)$: growth speed of particles of size x

$B(x)$: total fragmentation rate of particles of size x.

$k(y,x)$: probability that a particle of size y breaks and leaves a fragment of size x and one of size $y-x$ (we only consider binary fragmentation).

Why?

- Model widely used in science and technology (astrophysics, polymers, aerosols, grinding in mining industry, bacterial growth, etc...)

- Some history, mainly for pure fragmentation (N. K. Razumovsky, Dokl. Akad. Nauk SSSR, '40; A. N. Kolmogorov , Dokl. Akad. Nauk SSSR '41; V. S. Safronov's book on planets formation '69; S. K. Friedlander: Smoke, Dust & Haze, '76.)

- Recent ($\sim$'00$\rightarrow$) mathematic activity around: existence, uniqueness, spectral properties, qualitative behavior by different groups and different approaches. Strongly motivated by problems from biology.

- Very recent and interesting results and questions in the probabilistic literature.

Previous results & questions

PDE's methods. Typically if $B(x) = x^\gamma$, $\tau(x) = x^\alpha$ (although more general cases may be treated), under the condition

$$1 + \gamma - \alpha > 0 \tag{1}$$

The existence of first eigenvalue and eigenfunction is proved. Some examples of non existence are obtained when condition (1) is not fulfilled.
Exponential convergence to this eigenfunction asymptotically in time is also shown. Precise qualitative properties of these eigenfunctions have been obtianed. Special attention is paid to applications, in particular motivated by questions in biology.

Articles from $\sim$ 2000 by B. Perthame, L. Ryzhik, S. Mischler, P. Michel, P. Laurençot, M. Doumic, P. Gabriel, J. A. Cañizo, M. Cáceres, J. Scher...

Probabilistic methods. Probabilists are mostly interested in the stochastic growth fragmentation process, but also consider the equation.

Motivated also: problems of random geometry on spheres (J.F. Le Gall ICM'14).

They consider $B(x) = x^\gamma$ and $\tau(x) = x^\alpha$ too and kernels $k(y, x)$ like:

$$k(y,x) = \frac{1}{y} k_0 \left(\frac{x}{y} \right)$$

where k_0 is a non negative measure of compact support in $[0, 1]$ such that:

$$\int_0^1 x k_0(x) dx = 0, \;\; \int_0^1 k_0(x) dx > 1.$$

Articles from $\sim$ 2000 by J. Bertoin, B. Haas, G. Miermont J. Berestycki, A. R. Watson... In particular they like the case

$$1 + \gamma - \alpha = 0.$$

One first critical case.

$$\boxed{\gamma = 0, \;\; \alpha = 1, \;\; \text{so that } \; 1 + \gamma - \alpha = 0.}$$

This reduces to the fragmentation equation by the change: $u(t,x) = e^t v(t, xe^t)$ and gives:

$$\frac{\partial v}{\partial t}(t,y) + v(t,y) = \int_y^\infty \frac{1}{y} k_0\left(\frac{y}{z}\right) v(t,z)dz$$

This appears as critical case in the context of pure fragmentation equations:

$$\frac{\partial v}{\partial t}(t,y) + x^\gamma v(t,y) = \int_y^\infty \frac{1}{y} k_0\left(\frac{y}{z}\right) z^\gamma v(t,z)dz, \;\; \gamma \in \mathbb{R}.$$

Source solutions (initial data $u_0 = \delta(x-1)$) by E. D. McGrady & R. M. Ziff, '87.

J. Bertoin '03 (stoch. process for $\gamma = 0$); Mischler, R. Ricard & E. '05 ($\gamma > 0$); B. Haas '10 ($\gamma < 0$): long time behaviour, self similar solutions, ...

$\alpha = 1$, $\gamma = 0$, Results I.

(Joint work with M. Doumic.) The measure k_0 as before.

- If $u_0 \in L^1((1+x)dx)$, existence of a weak solution

$$u \in C([0,\infty); L^1((1+x)dx)) \cap L^\infty([0,\infty); L^1(xdx))$$

- All the functions:

$$u_s(t,x) := x^{-s} e^{(K(s)-1)t}, \text{ where } K(s) = \int_0^1 k_0(x) x^{s-1} ds$$

are pointwise solutions of the equation for all $s \in \mathbb{C}$ such that $K(s) < \infty$.

They are invariant under the scaling: $u(t,x) \mapsto u\left(t + \frac{s\log(\lambda)}{K(s)}, \lambda x\right), \lambda \in \mathbb{R}$.
But do not belong to $L^1(0,\infty))$ nor $L^1(x\,dx)$

Suppose not only: $$u_0 \geq 0, \quad \int\limits_0^\infty u_0(x)(1+x)dx < \infty \tag{2}$$

but also: $$u_0(x) = a_0 x^{-q_0} - a_0 q_0 x^{-q_0-1} + \mathcal{O}\left(x^r\right), \quad x \to \infty$$
$$u_0(x) = b_0 x^{-p_0} - b_0 p_0 x^{-p_0-1} + \mathcal{O}\left(x^\rho\right), \quad x \to 0,$$

for some $r > q_0$ and $\rho < p_0$ (by condition (2), $p_0 \leq 1$ and $q_0 \geq 2$).

- The unique solution $u \in C([0,\infty); L^1((1+x)dx)) \cap C^1((0,\infty); L^1((1+x)dx))$:

$$u(t,x) = \frac{1}{2\pi i} \int\limits_{\nu - i\infty}^{\nu + i\infty} U_0(s)\, e^{(K(s)-1)t} x^{-s} ds, \quad U_0(s) = \int_0^\infty u_0(x) x^{s-1} dx$$

for any $\nu \in (p_0, q_0)$.

This solution satisfies the conservation property:

$$\bullet \int_0^\infty xu(t,x)dx = \int_0^\infty xu_0(x)dx =: M\varphi(0), \quad \forall t > 0.$$

If $u \in C([0,\infty); L^1((1+x^{1+\varepsilon})dx)) \cap C^1((0,\infty); L^1((1+x^{1+\varepsilon})dx))$, $\varepsilon > 0$:

$$\bullet \ u(t) \rightharpoonup M\delta(0), \text{ as } t \to \infty.$$

- If: $$p_1 = \inf\left\{ s \in \mathbb{R};\ \ K(s) = \int_0^1 k_0(x)x^{s-1}dx < \infty \right\}$$

the function $K(s)$ is strictly convex on $(p_1, +\infty)$. It is not difficult to see that for any $t > 0$, $x \in (0,1)$ there exists a unique $s = s_+(t,x)$ such that

$$s_+(t,x) = (K')^{-1}\left(\frac{\log x}{t}\right).$$

Theorem 1

(i) $$u(t,x) = a_0\, x^{-q_0}\, e^{(K(q_0)-1)t}\left(1+\mathcal{O}\left(e^{(K(r-\delta)-K(q_0))t}\right)\right)$$

as $t\to\infty$, uniformly for all $x\geq 1; \forall\delta>0.$

(ii) $$u(t,x) = b_0\, x^{-p_0} e^{(K(p_0)-1)t}\left(1+\mathcal{O}\left(e^{-\frac{t\,K''(p_0)}{2}(s_+(t,x)-p_0)^2}\right)\right),$$

as $t\to\infty$, uniformly for all $x\in(0,1)$, $s_+(t,x)<p_0$,

(iii) $$u(t,x) = a_0\, x^{-q_0} e^{(K(q_0)-1)t}\left(1+\mathcal{O}\left(e^{-\frac{t\,K''(q_0)}{2}(s_+(t,x)-q_0)^2}\right)\right),$$

as $t\to\infty$, uniformly for all $x\in(0,1)$, $q_0<s_+(t,x)$,

The two functions $a_0\, x^{-q_0}\, e^{(K(q_0)-1)t}$ *and* $b_0\, x^{-p_0} e^{(K(p_0)-1)t}$ *are self-similar solutions of the equation. (*$q_0>0$ *and* $K(q_0)<1$*).*

The higher order terms are exponentially small uniformly for $x>1$*, or* $x\in(0,1)$ *and* $s_+(t,x)<p_0-\delta$ *or* $s_+(t,x)>q_0+\delta$ *for any* $\delta>0$ *arbitrarily small.*

Theorem 2. Suppose k_0 has a non zero absolutely continuous part.
Let $\varepsilon(t)$ be such that: $\lim_{t\to\infty} \varepsilon(t) = 0, \quad \lim_{t\to\infty} t\,\varepsilon^2(t) = \infty$.
For any $\delta > 0$ there exists two functions $\gamma_\delta(\cdot)$ and $\omega_\delta(\cdot)$ satisfying:

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\gamma_\delta(\varepsilon) = C_\delta \in \left(\frac{K''(q_0)}{2}, \frac{K''(p_0)}{2}\right), \qquad \lim_{\varepsilon\to 0} \omega_\delta(\varepsilon) = 0$$

and such that:

$$\boxed{u(t,x) = \Omega(t,x)^{-1}\left(U_0(s_+(t,x)) + \omega_\delta(\varepsilon(t)) + \mathcal{O}\left(e^{-t\gamma_\delta(\varepsilon(t))} + e^{-\frac{t\varepsilon^2(t)}{2}K''(q_0)}\right)\right)}$$

as $t \to \infty$, uniformly on $p_0 + \delta < s_+(t,x) < q_0 - \delta$ where

$$\Omega(t,x) = x^{s_+(t,x)} \frac{\sqrt{2\pi t K''(s_+(t,x))}}{e^{(K(s_+(t,x))-1)t}}.$$

$$u(t,x) = \Omega(t,x)^{-1}\left(U_0(s_+(t,x)) + \omega_\delta(\varepsilon(t)) + \mathcal{O}\left(e^{-t\gamma_\delta(\varepsilon(t))} + e^{-\frac{t\varepsilon^2(t)}{2}K''(q_0)}\right)\right)$$

By the properties of the function $\gamma_\delta(\varepsilon)$, we have

$$t\gamma_\delta(\varepsilon) = C_\delta t\varepsilon^2(t) + o\left(t\varepsilon^2(t)\right), \text{ as } t \to \infty$$

and the error term $\mathcal{O}(\cdot)$ decays exponentially fast in time.

- But, there exists kernels k_0, and data u_0, satisfying the hypothesis of Theorem 2 such that $\omega_\delta(\varepsilon(t))$ tends to zero at most algebraically as $t \to \infty$.

Conclusion: The asymptotic behavior of the solution depends on the pointwise behavior of the initial data (a_0, b_0, p_0, q_0). The rate of convergence may be algebraic.

The proofs are based on Mellin transform:

The Mellin transform.

Given a function u defined for $x \in (0, \infty)$:

$$M_u(t,s) = \int_0^\infty u(x) x^{s-1} dx.$$

If $u(t)$ solves the growth fragmentation equation with $B(x) = x^\gamma$ and $\tau(x) = x^\alpha$,

$$\frac{\partial}{\partial t} M_u(t,s) + \int_0^\infty \frac{\partial}{\partial x} \left(x^\alpha u(t,x) \right) x^{s-1} dx + \int_0^\infty u(t,x) x^{\gamma+s-1} =$$

$$= 2 \int_0^\infty \int_x^\infty \frac{1}{y} k_0 \left(\frac{x}{y} \right) y^\gamma u(t,y) dy \, x^{s-1} dx$$

$$= 2 \int_0^\infty u(t,y) y^{\gamma-1} dy \int_0^y k_0 \left(\frac{x}{y} \right) x^{s-1} dx$$

$$= 2 \int_0^\infty u(t,y) y^{s+\gamma-1} dy \int_0^1 k_0 \left(z \right) z^{s-1} dx$$

$$\frac{\partial}{\partial t}M_u(t,s)+\int_0^\infty \frac{\partial}{\partial x}\left(x^\alpha u(t,x)\right)x^{s-1}dx+\int_0^\infty u(t,x)x^{\gamma+s-1}=$$

$$=2\int_0^\infty u(t,y)y^{s+\gamma-1}dy\int_0^1 k_0\left(z\right)\ z^{s-1}dx$$

$$\Longrightarrow \frac{\partial}{\partial t}M_u(t,s)-(s-1)\int_0^\infty u(t,x)x^{\alpha+s-2}dx+M_u(t,s+\gamma)=$$

$$=2M_u(t,s+\gamma)\int_0^1 k_0\left(z\right)\ z^{s-1}dx$$

(if $x^{\alpha+s-1}u(t,x)\to 0$ as $x\to 0$ and $x\to\infty$).

$$\boxed{\frac{\partial}{\partial t}M_u(t,s)-(s-1)M_u(t,s+\alpha-1)=M_u(t,s+\gamma)(2K(s)-1)}$$

If $\gamma=0$ and $\alpha-1=0$ this is an ODE...

If $\gamma \neq 0$, $1+\gamma-\alpha=0$.

Considered in recent work by J. Bertoin and A. R. Watson (arXiv:'15).

They look for fundamental solutions in the general case $\gamma \in \mathbb{R}$, $1+\gamma-\alpha=0$. Part of their results may be briefly summarized as follows.

Under the condition:

$$\inf_{s\geq 0}\left(K(s+1)+s-1\right)<0$$

there exists a non negative, measure valued, solution of the equation in the sense of measures on $(0,\infty)$.

-The proofs for $\gamma<0$ use analytic arguments (Mellin transform).
-The authors can't use them for $\gamma>0$. Then, they build a positive self similar Markov processes such that the mass distribution of the particles solves the growth fragmentation equation.

More surprising

If

$$\inf_{s \geq 0} \left(K(s+1) + s - 1 \right) \geq 0$$

the authors exhibit an example of stochastic growth fragmentation process that blows up in finite time.

More precisely such that: for any $a > 0$, at some random time $t > 0$, there are infinitely many individuals in the compact set $[1, 1+a]$. (See also J. Bertoin & R. Stephenson arXiv:'16).

Their method does not give a solution to the growth fragmentation equation.

They ask then the question of the existence or not of solutions of the growth fragmentation equation.

When $\gamma \neq 0$

After Mellin transform:

$$\frac{\partial}{\partial t} M_u(t,s) = \Phi(s) M_u(t, s+\gamma)$$

$$M_u(0,s) = 1$$

$$\Phi(s) = (2K(s) + s - 2)$$

Step 1. Solve the auxiliary equation: $V(s+\gamma) = -\Phi(s)V(s)$

Step 2.

$$M_u(t,s) = \frac{1}{\gamma V(s)} \int_{\Re e\sigma = s_0} \frac{(-t)^{\frac{\sigma - s}{\gamma}} V(\sigma) d\sigma}{\Gamma\left(1 + \frac{\sigma - s}{\gamma}\right)\left(e^{-\frac{2\pi i}{\gamma}(\sigma - s)} - 1\right)}$$

Step 3. Invert the Mellin transform.

Satisfies formally the equation. Moreover, if $\Re e s > s_0$:

$$M_u(t,s) = \frac{1}{\gamma V(s)} \int_{\Re e\sigma = s_0} \frac{(-t)^{\frac{\sigma-s}{\gamma}} V(\sigma) d\sigma}{\Gamma\left(1+\frac{\sigma-s}{\gamma}\right)\left(e^{-\frac{2\pi i}{\gamma}(\sigma-s)}-1\right)}$$

$$= \frac{2i\pi}{\gamma} \underbrace{Res\left(\left(e^{-\frac{2\pi i}{\gamma}(\sigma-s)}-1\right)^{-1}; \sigma = s\right)}_{=-i\gamma/2\pi} +$$

$$+ \frac{1}{\gamma V(s)} \int_{\Re e\sigma = s_1} \frac{(-t)^{\frac{\sigma-s}{\gamma}} V(\sigma) d\sigma}{\Gamma\left(1+\frac{\sigma-s}{\gamma}\right)\left(e^{-\frac{2\pi i}{\gamma}(\sigma-s)}-1\right)}$$

$$= 1 + \frac{1}{\gamma V(s)} \int_{\Re e\sigma = s_1} \frac{(-t)^{\frac{\sigma-s}{\gamma}} V(\sigma) d\sigma}{\Gamma\left(1+\frac{\sigma-s}{\gamma}\right)\left(e^{-\frac{2\pi i}{\gamma}(\sigma-s)}-1\right)}$$

with $s_1 > \Re e s$. If all that makes sense... $\rightarrow$ Example:

Simplest case

Consider the homogeneous fragmentation kernel:

$$\begin{aligned}
k_0(x) &= 2H(x)H(1-x), \quad (H : \text{Heaviside's function}) \\
K(s) &= 2\int_0^1 x^{s-1}dx = \frac{2}{s}, \ \forall s > 0 \\
\inf_{s\geq 0}\left(K(s+1)+s-1\right) &= \inf_{s\geq 0}\left(\frac{2}{s+1}+s-1\right) \\
&\geq \inf_{s\geq 0}\left(\frac{2}{s}+s-2\right) = \frac{2}{\sqrt{2}}-1 > 0.
\end{aligned}$$

$$\begin{aligned}
\left(\frac{2}{s}+s-2\right) &= \frac{(s-\sigma_1)(s-\sigma_2)}{s} \\
\sigma_1 &= 1+i, \quad \sigma_2 = \overline{\sigma_1} = 1-i
\end{aligned}$$

Theorem. For all $\gamma > 0$ and $t > 0$ such that $\gamma t < 1$ let us define:

$$u(t,x) = (1-\gamma t)^{\frac{1}{\gamma}} \delta\left(x - (1-\gamma t)^{-\frac{1}{\gamma}}\right) + 2t(1-\gamma t)^{\frac{2}{\gamma}} \times$$
$$\times {}_2F_1\left(1+\frac{\sigma_1}{\gamma}, 1+\frac{\sigma_2}{\gamma}, 2, \gamma t\left(1+(\gamma t-1)x^\gamma\right)\right) H\left(1-(1-\gamma t)^{\frac{1}{\gamma}} x\right).$$

Then, u satisfies the growth fragmentation equation in the following weak sense:

$$\left\langle \frac{\partial u}{\partial t}, \varphi(t) \right\rangle = \mathcal{L}(u,\varphi)(t), \qquad \forall \varphi \in C_c^1\left(\left(0, \frac{1}{\gamma}\right) \times (0,\infty)\right);$$
$$\mathcal{L}(u,\varphi)(t) = \left\langle u(t), x^{\gamma+1}\frac{\partial \varphi}{\partial x}(t) \right\rangle - \langle u(t), x^\gamma \varphi(t)\rangle + 2\left\langle \int_x^\infty y^{\gamma-1} u(t,y)dy, \varphi(t,x) \right\rangle$$

(where $\langle \cdot, \cdot \rangle$ is the duality bracket between distributions of order one and C_c^1), and $u(t) \rightharpoonup \delta(x-1)$ as $t \to 0$ in the weak sense of measures.

Proof. The equation for the Mellin transform reads now:

$$\begin{aligned} \frac{\partial}{\partial t} M_u(t,s) &= \left(\frac{2}{s} + s - 2\right) M_u(t, s+\gamma) \\ &= \frac{(s-\sigma_1)(s-\sigma_2)}{s} M_u(t, s+\gamma) \\ \sigma_1 &= 1 + i, \quad \sigma_2 = \overline{\sigma_1} = 1 - i \end{aligned}$$

The equation for the auxiliary function V is:

$$V(s+\gamma) = -\left(\frac{2}{s} + s - 2\right) V(s).$$

A solution given by:

$$V(s) = \exp\left(-\gamma \int_{\Re e\sigma=\sigma_0} \log\left(-\Phi(\sigma)\right) \left(\frac{1}{1-e^{-2\gamma i(s-\sigma)}} - \frac{1}{1+e^{2\gamma i\sigma}}\right) d\sigma\right)$$

$$V(s) = (-\gamma)^{\frac{s}{\gamma}} \frac{\Gamma\left(\frac{s-\sigma_1}{\gamma}\right)\Gamma\left(\frac{s-\sigma_2}{\gamma}\right)}{\Gamma\left(\frac{s}{\gamma}\right)}$$

Plugging this in the expression for M_u gives...

$$M_u(t,s) = \Omega(t,s) = F\left(\frac{s-\sigma_1}{\gamma}, \frac{s-\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right)$$

where $F(a,b;c;z)$ denotes the Gauss hypergeometric function...
This could have been obtained form the beginning using that:

$$\frac{\partial}{\partial z} F(a,b;c;z) = \frac{ab}{c} F(a+1,b+1;c+1;z), \quad F(a,b;c;0) = 1$$

$$\frac{\partial}{\partial t} F\left(\frac{s-\sigma_1}{\gamma}, \frac{s-\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) = \gamma \frac{\frac{(s-\sigma_1)}{\gamma}\frac{(s-\sigma_2)}{\gamma}}{\frac{s}{\gamma}} F\left(\frac{s-\sigma_1}{\gamma}+1, \frac{s-\sigma_2}{\gamma}+1, \frac{s}{\gamma}+1, \gamma t\right)$$

$$= K(s) F\left(\frac{s+\gamma-\sigma_1}{\gamma}, \frac{s+\gamma-\sigma_2}{\gamma}, \frac{s+\gamma}{\gamma}, \gamma t\right).$$

When $|z| < 1$ and $-c \notin \mathbb{N}$

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)z^n}{\Gamma(c+n)\Gamma(n+1)}.$$

an absolutely convergent series.

Defines a meromorphic function of a, b, c, z for $|z| < 1$.

It may be extended by analyticity for values $|z| > 1$.

The point $z = 1$ is a ramification point. The principal branch has a cut along the real half line from 1 to ∞.

Another useful property: $F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c, z)$.

The solution may then be written as follows:

$$
\begin{aligned}
\Omega(t,s) &= (1-\gamma t)^{\frac{2-s}{\gamma}} F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) \\
&= (1-\gamma t)^{\frac{2-s}{\gamma}} \frac{\Gamma\left(\frac{s}{\gamma}\right)}{\Gamma\left(\frac{\sigma_1}{\gamma}\right)\Gamma\left(\frac{\sigma_2}{\gamma}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\sigma_1}{\gamma}+n\right)\Gamma\left(\frac{\sigma_2}{\gamma}+n\right)(\gamma t)^n}{\Gamma\left(\frac{s}{\gamma}+n\right)\Gamma(n+1)}
\end{aligned}
$$

for $|\gamma t| < 1$.

• For all γ and t given such that $|\gamma t| < 1$, the solution $\Omega(t,s)$ is a meromorphic function of s with poles located at $s = -m\gamma$, $m = 0, 1, 2, \ldots$

• The function $\Omega(t,s)$ has analytic extensions to the whole complex plane. In particular to regions where $|\gamma t| > 1$.

Inverse Mellin of Ω.

What is the inverse Mellin transform of $\Omega(t,s)$?

Lemma. Suppose $\sigma_1 \in \mathbb{C}$, $\sigma_2 \in \mathbb{C}$, $\gamma > 0$ and $t > 0$ such that $1 - \gamma t > 0$ and define the function

$$v(t,x) = F\left(1 + \frac{\sigma_1}{\gamma}, 1 + \frac{\sigma_2}{\gamma}, 2, \gamma t\left(1 + (\gamma t - 1)x^\gamma\right)\right) H\left(1 - (1 - \gamma t)^{\frac{1}{\gamma}} x\right)$$

for $x > 0$, where H is the Heaviside function. Then, for all $s > 0$, the Mellin transform of v is given by:

$$M_v(t,s) \equiv \int_0^\infty v(t,x) x^{s-1} dx = \frac{(1-\gamma t)^{-\frac{s}{\gamma}}}{\sigma_1 \sigma_2 t} \left(F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) - 1\right).$$

Proof. If $0 < \gamma t < 1$ and $(1-\gamma t)^{1/\gamma} x < 1$ it follows $\gamma t\,(1 + (\gamma t - 1) x^{\gamma}) \in (0,1)$

- $$\int_0^{\infty} v(t,x) x^{s-1} dx = \int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}} v(t,x) x^{s-1} dx$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(1+\frac{\sigma_1}{\gamma}+n\right)\Gamma\left(1+\frac{\sigma_2}{\gamma}+n\right)(\gamma t)^n}{\Gamma\left(1+\frac{\sigma_1}{\gamma}\right)\Gamma\left(1+\frac{\sigma_2}{\gamma}\right)\Gamma(2+n)\,\Gamma(n+1)} \int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}} (1+(\gamma t-1)x^{\gamma})^n x^{s-1} dx$$

- $$\int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}} (1+(\gamma t-1)x^{\gamma})^n x^{s-1} dx = \sum_{m=0}^{n} \binom{n}{m} (-1)^m (1-\gamma t)^m \int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}} x^{m\gamma+s-1} dx$$
$$= (1-\gamma t)^{-\frac{s}{\gamma}} \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{m\gamma+s} = (1-\gamma t)^{-\frac{s}{\gamma}} \frac{\Gamma(n+1)\Gamma\left(\frac{s}{\gamma}\right)}{\gamma\Gamma\left(1+\frac{s}{\gamma}+n\right)}.$$

Use the second expression into the first one, the result follows.

It follows from the previous Lemma that if

$$u(t,x) = (1-\gamma t)^{\frac{1}{\gamma}}\delta\left(x-(1-\gamma t)^{-\frac{1}{\gamma}}\right) + 2t(1-\gamma t)^{\frac{2}{\gamma}} \times$$
$$\times F\left(1+\frac{\sigma_1}{\gamma}, 1+\frac{\sigma_2}{\gamma}, 2, \gamma t\left(1+(\gamma t-1)x^{\gamma}\right)\right) H\left(1-(1-\gamma t)^{\frac{1}{\gamma}}x\right)$$

then, $M_u(t,s) = \Omega(t,s)$.

Moreover, the inverse Mellin transform of $\Omega(t)$, $M_\Omega^{-1}(t,x)$, is well defined and

$$M_\Omega^{-1}(t,x) = \frac{1}{2i\pi}\int_{\Re e s = s_0} \Omega(t,s)x^{-s}ds = u(t,x), \;\; \forall s_0 > 0$$

is well defined for all $x > 0$ because, by well known properties of F:

$$\left|\Omega(t,s) - (1-\gamma t)^{\frac{2-s}{\gamma}}\left(1+\frac{2t}{s}\right)\right| \le C(1-\gamma t)^{\frac{2-s}{\gamma}}(1+|s|)^{-3}.$$

Multiply the equation for $\Omega(t,s)$ by x^{-s} and integrate over $\Re e s = s_0$ for $s_0 > 0$:

$$\frac{\partial}{\partial t}\int_{\Re e s = s_0} \Omega(t,s)x^{-s} = \int_{\Re e s = s_0} \frac{(s-\sigma_1)(s-\sigma_2)}{s}\Omega(t,s+\gamma)x^{-s}ds$$

$$\equiv \int_{\Re e s = s_0} \left(\frac{2}{s} + (s-1) - 1\right)\Omega(t,s+\gamma)x^{-s}ds$$

$$\int_{\Re e s = s_0} \Omega(t,s)x^{-s} = 2i\pi u(t,x).$$

$$\int_{\Re e s = s_0} \Omega(t,s+\gamma)x^{-s}ds = x^{\gamma}\int_{\Re e \sigma = s_0+\gamma} \Omega(t,\sigma)x^{-\sigma}ds.$$

Since $\Omega(t)$ is analytic on $\Re e s > 0$, $\Omega(t,s) \to 0$ as $|s| \to \infty$ and $s_0 + \gamma > s_0 > 0$:

$$\int_{\Re e \sigma = s_0+\gamma} \Omega(t,\sigma)x^{-\sigma}ds = \int_{\Re e \sigma = s_0} \Omega(t,\sigma)x^{-\sigma}ds = 2i\pi u(t,x).$$

Arguing similarly for the remaining terms:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(x^{\gamma+1}u\right) + x^{\gamma}u = 2\int_x^{\infty} y^{\gamma-1}u(t,y)dy$$

where the equation is satisfied in the space of distributions of order -1.

- From the explicit expression of u: $u(t) \rightharpoonup \delta(x-1)$ as $t \to 0$.
- The measure u is non negative:

$$u(t,x) = (1-\gamma t)^{\frac{1}{\gamma}}\delta\left(x-(1-\gamma t)^{-\frac{1}{\gamma}}\right) + 2t(1-\gamma t)^{\frac{2}{\gamma}} \times$$
$$\times F\left(1+\frac{\sigma_1}{\gamma}, 1+\frac{\sigma_2}{\gamma}, 2, \gamma t\left(1+(\gamma t-1)x^{\gamma}\right)\right) H\left(1-(1-\gamma t)^{\frac{1}{\gamma}}x\right)$$

(using that $\sigma_2 = \overline{\sigma_1}$).

- All positive moments of u are finite and known $\equiv \Omega(t,s)$.

This may be improved to the following.

Proposition. The function

$$u_R(t,x) = u(t,x) - (1-\gamma t)^{\frac{2-1}{\gamma}} \delta\left(x - (1-\gamma t)^{-\frac{1}{\gamma}}\right)$$

satisfies the equation:

$$\frac{\partial u_R}{\partial t}(t,x) + \frac{\partial}{\partial x}(x^{\gamma+1} u_R)(t,x) + x^\gamma u_R(t,x) = 2\int_{[0,\infty)} y^{\gamma-1} u(t,y)dy$$

for all $x > 0$, $t \in (0, \gamma^{-1})$. The measure $u_s = (1-\gamma t)^{\frac{\sigma_1+\sigma_2-1}{\gamma}} \delta\left(x - (1-\gamma t)^{-\frac{1}{\gamma}}\right)$ satisfies

$$\frac{\partial u_s}{\partial t} + \frac{\partial}{\partial x}\left(x^{\gamma+1} u_s\right) + x^\gamma u_s = 0, \quad \text{in } \mathcal{D}'_{-1}\left((0,\gamma^{-1}) \times (0,\infty)\right).$$

Proof. Let us check for example the second assertion:

$$\frac{\partial u_s}{\partial t} = -(1-\gamma t)^{\frac{1}{\gamma}-1}\delta\left(x-(1-\gamma t)^{-\frac{1}{\gamma}}\right) - (1-\gamma t)^{-1}\delta'\left(x-(1-\gamma t)^{-\frac{1}{\gamma}}\right)$$

$$\frac{\partial}{\partial x}\left(x^{\gamma+1}u_s\right) = \frac{\partial}{\partial x}\left((1-\gamma t)^{-\frac{\gamma+1}{\gamma}}u_s\right) = (1-\gamma t)^{-\frac{\gamma+1}{\gamma}}\frac{\partial u_s}{\partial x}$$

$$= (1-\gamma t)^{-\frac{\gamma+1}{\gamma}+\frac{1}{\gamma}}\delta'\left(x-(1-\gamma t)^{-\frac{1}{\gamma}}\right)$$

$$x^\gamma u_s = x^\gamma(1-\gamma t)^{\frac{1}{\gamma}}\delta\left(x-(1-\gamma t)^{-\frac{1}{\gamma}}\right)$$

$$= (1-\gamma t)^{-1}(1-\gamma t)^{\frac{1}{\gamma}}\delta\left(x-(1-\gamma t)^{-\frac{1}{\gamma}}\right)$$

Limit as $\gamma t \to 1^-$

The limit of $u(t)$ as $\gamma t \to 1^-$ easily follows from the limit of $F(a,b;c;z)$ as $z \to 1^-$. This depends on the values of a, b, c. In our case we have: $\Re e(c-a-b) = -\frac{\sigma_1+\sigma_2}{\gamma} = -\frac{2}{\gamma} < 0$ and then:

$$\lim_{\gamma t \to 1^-} u_R(t,x)\,(1-\gamma t)^{-\frac{2-(\sigma_1+\sigma_2)}{\gamma}}\,(1+\gamma t x^\gamma)^{\frac{\sigma_1+\sigma_2}{\gamma}} = \frac{\Gamma\left(\frac{\sigma_1+\sigma_2}{\gamma}\right)\sigma_1\sigma_2}{\gamma\Gamma\left(1+\frac{\sigma_1}{\gamma}\right)\Gamma\left(1+\frac{\sigma_2}{\gamma}\right)} =: C$$

$$\boxed{u_R(t,x) \sim C(1+x^\gamma)^{-\frac{\sigma_1+\sigma_2}{\gamma}}, \;\; \gamma t \to 1^-}$$

Remark. For all $\gamma t < 1$ the measure u is such that

$$
\begin{aligned}
M_u(t,s) &= \int_0^\infty u(t,x)x^{s-1}ds < \infty, \quad \forall s > 0 \\
&= (1-\gamma t)^{\frac{2-s}{\gamma}} F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right)
\end{aligned}
$$

but as $\gamma t \to 1$

$$
\int_0^\infty u(t,x)x^{s-1}ds \sim \int_0^\infty (1+x^\gamma)^{-\frac{\sigma_1+\sigma_2}{\gamma}} x^{s-1} ds = 0, \text{ if } s \geq 2.
$$

A first natural question now is the extension of this solutions to times $t > 1/\gamma$. Use of the analytic extensions of $\Omega(t,s)$ for $|\gamma t| > 1$?