Regularity of the Boltzmann Equation in Bounded Domains

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Introduction

The Boltzmann equation

$F(t,x,v)$: density distribution (in space $x \in \Omega \subset \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$) of gas particles at time t.

$$\underbrace{\partial_t F + v \cdot \nabla_x F}_{\text{free transport}} = \underbrace{Q(F,F)}_{\text{collisions}},$$

$$\begin{aligned} Q(F_1,F_2) &= Q_{gain}(F_1,F_2) - Q_{loss}(F_1,F_2) \\ &= \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |v-u|^{\kappa} \{F_1(u')F_2(v') - F_1(u)F_2(v)\} q_0 \, \mathrm{d}\omega \, \mathrm{d}u, \end{aligned}$$

where $v' = v + [(u-v)\cdot\omega]\,\omega,$
$u' = u - [(u-v)\cdot\omega]\,\omega.$

Assume $0 < \kappa \leq 1$ (hard potential),
$0 \leq q_0 \leq C \frac{v-u}{|v-u|} \cdot \omega$ (angular cutoff).

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The Maxwellian density $\mu(v) = \exp\{-v^2/2\}$ verifies $Q(\mu, \mu) = 0$.

Boundary conditions

Ω is a bounded domain of $\mathbb{R}^3$.
We define in the phase space the outgoing/incoming/grazing boundary

$$\gamma_+ = \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : v \cdot n(x) > 0\},$$
$$\gamma_- = \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : v \cdot n(x) < 0\},$$
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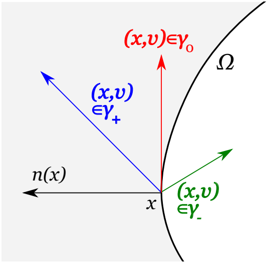

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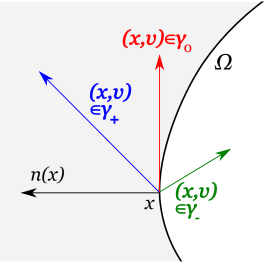

Diffuse boundary condition:

$$\forall (x,v) \in \gamma_-, \quad F(t,x,v) = c_\mu\, \mu(v) \int_{u \cdot n(x) > 0} F(t,x,u)\, u \cdot n(x)\, \mathrm{d}u.$$

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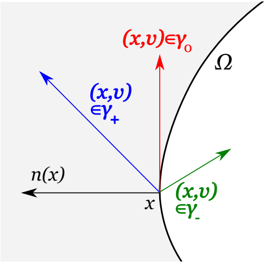

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Remark: $\mu(v) = \exp\{-v^2/2\}$ is an equilibrium of the system.

Known results in general bounded domains

Existence of renormalized solutions (DiPerna-Lions type):
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Existence of global strong solutions?
In a perturbative framework (following Ukai, Asano, Guiraud, ...).

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Perturbative framework: $F \sim \mu(v) = e^{-|v|^2/2}$.

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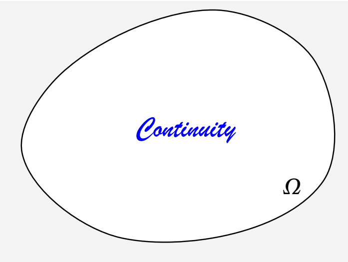

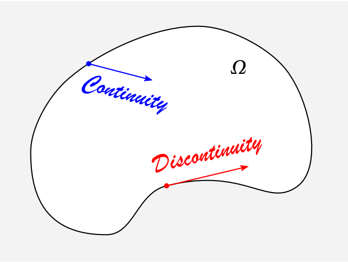

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Theorem (Existence/uniqueness, [Guo 10], [GKTT 13])

Let $0 < \theta' < \theta < 1/4$. Let $f_0 \geq 0$ in $L^{\infty,\theta}_{x,v}$. There exists a unique solution $F = \sqrt{\mu}\, f \geq 0$ of the Boltzmann equation with diffuse BC on $[0, T^*)$ with $T^* = T^*(f_0)$. Furthermore $\forall T < T^*$,

$$\sup_{0\leq t\leq T} \|e^{\theta'|v|^2} f(t)\|_\infty \lesssim_T \|e^{\theta|v|^2} f_0\|_\infty.$$

If $\|e^{\theta|v|^2}\{f_0 - \sqrt{\mu}\}\|_\infty \ll 1$, then $T^* = \infty$.

Sobolev regularity in convex domains [GKTT 13]

Assume Ω is a smooth strictly convex domain.
Assume $f_0 \geq 0$ in $L^{\infty,\theta}_{x,v}$ and f_0 satisfies the diffuse boundary condition.

Theorem ($W^{1,p}$ propagation, $1 < p < 2$)

Assume $p \in (1,2)$ and $\nabla f_0 \in L^p(\Omega \times \mathbb{R}^3)$. Then

$$\forall T < T^*, \sup_{0 \leq t \leq T} \|\nabla f\|_p^p(t) + \int_0^T \iint_{\partial\Omega\times\mathbb{R}^3} |\nabla f|^p |v.n| \,\mathrm{d}S\,\mathrm{d}v\,\mathrm{d}t$$
$$\lesssim_T \|\nabla f_0\|_p^p + P(\|e^{\theta|v|^2} f_0\|_\infty),$$

with P a polynomial.

Note: we use the notations $\nabla = [\partial_t, \nabla_x, \nabla_v]$, and
$\partial_t f_0 := -v \cdot \nabla_x f_0 + \sqrt{\mu}^{-1} Q(\sqrt{\mu} f_0, \sqrt{\mu} f_0)$.

Proof: entropy estimates, Grad's estimates, Ukai's lemma associated with a specific treatment of the almost grazing boundary.

Singularity at the grazing boundary

Formally, on the boundary

$$\partial_n f_{|\gamma} = \frac{1}{v \cdot n} \left\{ -\partial_t f - \sum (v \cdot \tau_i)\, \partial_{\tau_i} f + \sqrt{\mu}^{-1} Q(\sqrt{\mu} f, \sqrt{\mu} f) \right\}.$$

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We integrate on the incoming boundary

$$\int_{\gamma_-} |\partial_n f|^p |v \cdot n| \, \mathrm{d}S_x \, \mathrm{d}v \lesssim \underbrace{\int_{\mathbb{R}^3} |v \cdot n|^{1-p}}_{<\infty \text{ when } p<2}.$$

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▶ We define a kinetic distance $\alpha = \alpha(x, v)$ which balances the singularity.

Kinetic distance

The (strictly convex) domain Ω is defined by

$$\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}.$$

Define the kinetic distance

$$\alpha(x, v) := |v \cdot \nabla_x \xi|^2 - 2\{v \nabla_x^2 \xi v\}\xi(x) \geq 0.$$

Main features:

- "distance": vanishes exactly on the grazing boundary
- "kinetic": invariant along the characteristics (up to some quantity in $|v|$).

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- "distance": vanishes exactly on the grazing boundary
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We therefore consider, for $\omega \gg 1$,

$$d_p^2(t, x, v) := e^{-\omega t \sqrt{1+v^2}} \alpha(x, v)^{\beta_p}.$$

Weighted Sobolev regularity in convex domains [GKTT 13]

Assume Ω is a smooth strictly convex domain.
Assume $f_0 \geq 0$ in $L^{\infty,\theta}_{x,v}$ and f_0 satisfies the diffuse boundary condition.

Theorem (Weighted $W^{1,p}$ propagation, $p \in [2,\infty]$)

Assume $d_p^2 \nabla f_0 \in L^p(\Omega \times \mathbb{R}^3)$. Then $\forall T < T^*$, if $p \in [2,\infty[$

$$\sup_{0 \leq t \leq T} \|d_p^2 \nabla f\|_p^p(t) + \int_0^T \iint_{\partial\Omega \times \mathbb{R}^3} |d_p^2 \nabla f|^p |v.n| \, \mathrm{d}S_x \, \mathrm{d}v \, \mathrm{d}t \lesssim_T \|d_p^2 \nabla f_0\|_p^p + P(\|e^{\theta|v|^2} f_0\|_\infty),$$

and if $p = \infty$, $\sup \|d_p^2 \nabla f\|_\infty(t) \lesssim_T \|d_p^2 \nabla f_0\|_\infty + P(\|e^{\theta|v|^2} f_0\|_\infty)$,
where $d_p^2 = d_p^2(t,x,v) \geq 0$ vanishes on the grazing boundary γ_0.

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Theorem (C^1 propagation)

If $d_\infty^2 \nabla f_0 \in C^0(\bar{\Omega} \times \mathbb{R}^3)$ and $\partial_t f_0$ satisfies the diffuse boundary condition, then f is C^1 away from the grazing boundary.

Introduction

Convex domain

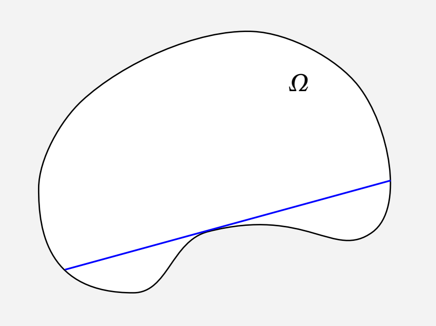
Non-convex domains

Conclusion

Grazing trajectories

- We get rid of the grazing trajectories

$$\mathfrak{S}_{\mathrm{B}} := \{(x,v) \in \bar{\Omega} \times \mathbb{R}^3 : n(x_{\mathbf{b}}(x,v)) \cdot v = 0\}$$



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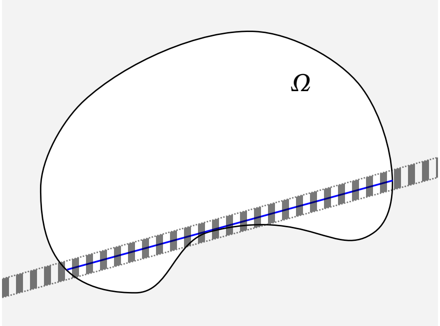
by applying a cut-off on a tubular neighborhood of $\mathfrak{S}_{\mathrm{B}}$.



BV regularity in non-convex domains [GKTT 14]

Assume Ω is a smooth bounded domain.
Assume $f_0 \geq 0$ in $L^{\infty,\theta}_{x,v}$.

Theorem (BV propagation)

If $f_0 \in BV(\Omega \times \mathbb{R}^3)$, then $\forall T < T^*$,

$$\sup_{0 \leq t \leq T} ||f(t)||_{BV} \lesssim_T ||f_0||_{BV} + P(||e^{\theta|v|^2} f_0||_\infty),$$

and $\nabla_{x,v} f_\gamma$ is a Radon measure $\sigma(t)$ on $\partial\Omega \times \mathbb{R}^3$ such that

$$\int_0^T |\sigma(\partial\Omega \times \mathbb{R}^3)| \, \mathrm{d}t \lesssim_T ||f_0||_{BV} + P(||e^{\theta|v|^2} f_0||_\infty).$$

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Recap and optimality of results

In convex domains	In non-convex domains
C^0 away from γ_0 [Guo 10] $W^{1,p}$ for $1 \leq p < 2$ ~~H^1~~: Ct-ex. (transport eq.) Weighted $W^{1,p}$ for $p \in [2, \infty]$ C^1 away from γ_0 ~~$W^{2,1}$~~: Ct-ex.	~~C^0~~: Discontinuity created on γ_0 and propagated along the grazing trajectories [Kim 11] BV propagation

Further results and questions

Other boundary conditions

Specular boundary conditions: in convex domains, propagation of C^1 regularity away from γ_0 (with the help of the kinetic distance) [GKTT13]
Bounce-back boundary conditions: same [GKTT13], in non-convex domains: propagation of discontinuity [Kim 11]
Maxwell boundary condition: continuity away from the grazing trajectories [Briant Guo 15].

Non-isothermal boundary

Results of existence of strong solutions, uniqueness and stability (exponential convergence towards the solution of the stationnary problem) for a (*not too much*) varying boundary temperature.
Continuity propagation in convex domains [Esposito Guo Kim Marra 13].

Thank you.