Hydrodynamic Limits of Granular Gases to Pressureless Euler in Dimension 1

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Outline of the Talk

1. Generalities on the Granular Gases Equation
   - Introduction
   - Modeling

2. Macroscopic Descriptions

3. Pressureless Euler Limit
   - Mathematical Results
   - The functional $L_{\eta,\mu,k}$
The Granular Gases Equation

The granular gases equation describes the behavior of a dilute gas of particles when the only interactions taken into account are binary *inelastic* collisions.

Applications

Pollen dissemination, avalanches, planetary rings, ...

\[1\] Courtesy of T. Pöschel
The clustering phenomenon

Numerical method. Rescaling velocity method, using some qualitative properties of the equation\(^2\).

Results. Local density, \(\varepsilon = 0.05\): granular gas (left) vs. perfect elastic gas (right), at time \(T = 2\).

Numerical parameters. \(2d_x \times 3d_v\) model, \(N_x = N_y = 100, N_{vx} = N_{vy} = N_{vz} = 32\).

\[\rightarrow\] Consistent with experiments (Brillantov, Pöschel, 2004).

\(^2\)F. Filbet, TR, \(JCP\) (2013)
A Boltzmann-like Kinetic Equation

General Scaled Equation

Study of a particle distribution function $f_\varepsilon(t, x, v)$, depending on time $t > 0$, space $x \in \Omega \subset \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$, solution to

$$
\begin{aligned}
\frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon &= \frac{1}{\varepsilon} Q_e(f_\varepsilon, f_\varepsilon), \\
f_\varepsilon(0, x, v) &= f_0(x, v),
\end{aligned}
$$

(1)

where $Q_e$ is the inelastic collision operator, describing the microscopic collision dynamic and $\varepsilon$ is a scaling parameter.

$\varepsilon$ is the Knudsen number, ratio of the mean free path between collision by the typical length scale of the problem;

$Q_e$ only acts on the $v$ variable;

Boundary conditions in space needed to describe completely the problem.
Modeling Assumptions

Microscopic dynamics

- **Localized interactions**: particles interact only by contact (collision), at a given time $t$ and a given position $x$;
- **Diluted gases**: collisions occur between two particles at the same time (we neglect the collisions of three particles or more);
- **Micro-reversible collisions**: the collision dynamics is time-reversible (at the microscopic level);
- **Boltzmann chaos assumption**: the velocity of two colliding particles are uncorrelated before collision.

The microscopic collision process is said to be

- **elastic** when the kinetic energy is conserved during a collision (this is for example the case for a perfect molecular gas);
- **inelastic** when a fraction of the kinetic energy is dissipated during a collision (this is the case of a granular gas).
Inelastic Hard Spheres

- Microscopic dynamics: conservation of impulsion and dissipation of kinetic energy;
- Constant restitution coefficient \( e \in [0, 1] \);
- Parametrization of the post-collisional velocities \((v', v'_*)\) as a function of the pre-collisional velocities \((v, v_*)\):

\[
\begin{align*}
v' &= v - \frac{1 + e}{2} ((v - v_*) \cdot \omega) \omega, \\
v'_* &= v_* + \frac{1 + e}{2} ((v - v_*) \cdot \omega) \omega,
\end{align*}
\]

where \( \omega \in S^{d-1} \) is the impact direction;

(green is elastic, red is inelastic)
Features of the Model

If $e < 1$ (true inelasticity),

- **Trivial equilibria** (Dirac masses) because of the temperature dissipation: linearized study difficult;
- Particle velocities highly **correlated**: non Gaussian tails;
- Mathematical analysis possible only in the $L^1$ Banach setting: no scalar products, perturbative theory intricate;
- No formal **entropy dissipation** structure\(^3\): very few energy estimates available, large time behavior difficult to investigate;
- Macroscopic description formally given by the **pressureless gas dynamics**: creation of singularities in finite time;
- Creation of spatial d’**inhomogeneities** and zero temperature zones (clustering phenomenon).
- Meaningful even in dimension **1**: the collision process is given by

$$\{v', v'_*\} = \{v, v_*\} \quad \text{or} \quad \left\{ \frac{v + v_*}{2} \pm \frac{e}{2}(v - v_*) \right\}.$$ 

\(^3\)Nevertheless, extensive numerical experiments show that the Boltzmann entropy would be a good candidate, see Garcia, Maynar, Mischler, Mouhot, TR, Trizac, *JSM* (2015)
The Inelastic Collision Operator

For two velocities $v, v* \in \mathbb{R}^d$, we set $u := v - v*$, $\hat{u} := \frac{u}{|u|}$; if $\psi$ is a smooth test function, $\psi' := \psi(v')$, $\psi* := \psi(v*)$, $\psi'_* := \psi(v'_*)$.

Representations of the collision operator

- **Weak form** of the collision operator:
  \[
  \int_{\mathbb{R}^d} Q_e(f, f)(v) \psi(v) \, dv = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}} |u| f f' \left( \psi' + \psi'_* - \psi - \psi* \right) b(\hat{u} \cdot \sigma) \, d\sigma \, dv \, dv*,
  \]

  where $b \geq 0$ the collisional cross section.

- **Strong form**: if $e$ is non-zero,
  \[
  Q_e(f, f)(v) = \int_{\mathbb{R}^d \times S^{d-1}} |u| \left( \frac{1}{e^2} f' f'_* - f f_\* \right) b(\hat{u} \cdot \sigma) \, d\sigma \, dv. \]
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Macroscopic Properties of the Operator

Conservation of mass and momentum, dissipation of kinetic energy:

\[
\int_{\mathbb{R}^d} Q_e(f, f)(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \\ -(1 - e^2) D(f, f) \end{pmatrix}
\]

where \( D(f, f) \geq 0 \) is the energy dissipation functional:

\[
D(f, f) := b_1 \int_{\mathbb{R}^d \times \mathbb{R}^d} f f_* |v - v_*|^3 \, dv \, dv_* \leq b_1 \rho^{5/2} \left( \int_{\mathbb{R}} f(v) |v - u|^2 \, dv \right)^{\frac{3}{2}} \geq 0.
\]

and \( b_1 \) is the angular momentum:

\[
b_1 := \int_{S^{d-1}} (1 - (\hat{u} \cdot \omega)) b(\hat{u} \cdot \omega) \, d\omega < \infty.
\]
Macroscopic Properties of the Operator

Conservation of mass and momentum, dissipation of kinetic energy:

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where \( D(f,f) \geq 0 \) is the energy dissipation functional:

\[
D(f,f) := b_1 \int_{\mathbb{R}^d \times \mathbb{R}^d} ff^* |v - v^*|^3 dv dv^*,
\]

\[
\geq b_1 \rho^{5/2} \left( \int_{\mathbb{R}} f(v) |v - u|^2 dv \right)^{3/2} \geq 0.
\]

and \( b_1 \) is the angular momentum:

\[
b_1 := \int_{\mathbb{S}^{d-1}} (1 - (\hat{u} \cdot \omega)) b(\hat{u} \cdot \omega) d\omega < \infty.
\]

Microscopic consequence: Concentration in the velocity space!
Monokinetic Distribution and Pressureless Dynamics

Define the first moments of $f$ as

$$(\rho, \rho u, E) = \int_{\mathbb{R}^d} f(v) \varphi(v) dv, \quad \varphi(v) = (1, v, |v|^2/2).$$

Dissipation of energy formally yields that when $\varepsilon \to 0$,

$$f(t, x, v) \xrightarrow{\varepsilon \to 0} \rho(x) \delta_0 (v - u(x)), \quad \forall (x, v) \in \Omega \times \mathbb{R}^d,$$

where, using the macroscopic properties of $Q_e$,

$$
\begin{cases}
\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) = 0.
\end{cases}
$$
Monokinetic Distribution and Pressureless Dynamics

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where, using the macroscopic properties of $Q_e$,

$$\begin{cases} 
\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) = 0.
\end{cases}$$

**Pressureless Euler/Sticky particles equation:** Generation of $\delta$-singularity in finite time and transient clusters formation.
Quasi-Elastic Limit and Compressible Euler Dynamics

One can show\(^4\) that if \(1 - e \sim \varepsilon\), then

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{Q_1(f, f)}{\varepsilon} + \frac{1 - e}{\varepsilon} \mathcal{I}(f, f) + \mathcal{O}\left(\frac{(1 - e)^2}{\varepsilon}\right).
\]

for an explicit friction operator \(\mathcal{I}\).

When \(\varepsilon \to 0\), there is a compressible Euler-like dynamics\(^5\)

\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho(u \otimes u + TI)) &= 0, \\
\partial_t E + \nabla_x \cdot (u(E + \rho T)) &= -C_d \rho^2 T^{3/2},
\end{align*}
\]

which also exhibits cluster formation\(^6!\)

\(^4\)G. Toscani (2004)
\(^5\)The closure is made thanks to the Maxwellian

\[
\mathcal{M}_{\rho, u, T}(v) = \frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{|v - u|^2}{2T}\right).
\]

\(^6\)Carrillo, Salueña (2006)
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A State of the Art on the Granular Gases Equation

Space homogeneous setting \( \partial_t f = Q_e(f, f) \)

- **Cauchy problem** for different types of kernels: Bobylev-Carrillo-Gamba (2000), Bobylev-Cercignani-Toscani (2003), Mischler-Mouhot-Ricard (2006);
- **Qualitative behavior**: Bobylev-Gamba-Panferov (2004), Gamba-Panferov-Villani (2004);

Full space-dependent equation \( \partial_t f + v \cdot \nabla_x f = Q_e(f, f) \)

- **Cauchy problem** 1d: Benedetto-Caglioti-Pulvirenti (1997, 2001);
- **Cauchy problem** near vacuum: Alonso (2009);
- **Cauchy problem** quasi-homogeneous, in the torus, thermal bath: Tristani (2013);
A State of the Art on the Pressureless Euler Equation

\[
\begin{aligned}
\begin{cases}
\partial_t \rho + \nabla_x \cdot (\rho \, u) = 0, \\
\partial_t (\rho \, u) + \nabla_x \cdot (\rho \, u \otimes u) = 0.
\end{cases}
\end{aligned}
\]

Attract a lot of interest from the physics community because of its description of the large scale structure of the universe: Silk-Szalay-Zeldovich (1983)

- A rather delicate equation. Exhibits shocks because \( u \) (formally) solves Burgers equation where \( \rho > 0 \Rightarrow \) concentration in \( \rho \Rightarrow \) ill-posed system!

- Nevertheless, wellposedness theory exists by imposing semi-Lipschitz condition on \( u \): Bouchut-James (1999), Boudin (2000), Huang-Wang (2001-2004);

- Model obtained as the hydrodynamic limit of the sticky particle model
  - In dimension 1: E-Rykov-Sinai (1996), Brenier-Grenier (1998),
One Dimensional Sticky Particles

→ **Known results.** Hydrodynamic limit of the one dimensional $N$–particles system with sticky collisions\(^7\) towards the *pressureless Euler* equations

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho \ u) &= 0, \\
\partial_t (\rho \ u) + \partial_x (\rho \ u^2) &= 0.
\end{align*}
\]

(2)

→ **Method.** Convergence of the empirical density towards a monokinetic distribution, using (among other arguments) the (microscopic) Oleinik property

\[
\sup_{i \in \{1, \ldots, N\}} \frac{(v_{i+1} - v_i)_+}{(x_{i+1} - x_i)_+} < \frac{1}{t},
\]

where $(x_i, v_i)$ represents the position and velocity of the $i^{th}$ particle at time $t$.

\(^7\)after collisions, the two particles stick together and travel with their average velocity.
One Dimensional Sticky Particles

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\]

where $((x_i, v_i))$ represents the position and velocity of the $i^{th}$ particle at time $t$.

**Goal.**

Convergence of the particle distribution function $f(t, x, v)$ solution to the granular gases equation towards the monokinetic density

\[
f(t, x, v) = \rho(t, x) \delta(x - u(t, x)),
\]

where $(\rho, u)$ solution to the pressureless Euler system (2).

\(^7\) after collisions, the two particles stick together and travel with their average velocity.
Hydrodynamic Limit of the Granular Gases Equation

Theorem (P.-E. Jabin, TR, 2016)

Consider a sequence of weak solutions \( f_\varepsilon(t, x, v) \in L^\infty([0, T], L^p(\mathbb{R}^2)) \) for some \( p > 2 \) and with total mass 1 to the granular gases Eq. (1) such that for any \( k \)

\[
\sup_\varepsilon \int_{\mathbb{R}^2} |v|^k f_\varepsilon^0(x, v) \, dx \, dv < \infty, \quad \sup_\varepsilon \int_{\mathbb{R}^2} |x|^2 f_\varepsilon^0(x, v) \, dx \, dv < \infty,
\]

and \( f_\varepsilon^0 \) is, uniformly in \( \varepsilon \), in some \( L^p \) for \( p > 1 \)

\[
\sup_\varepsilon \int_{\mathbb{R}^2} (f_\varepsilon^0(x, v))^p \, dx \, dv < \infty. \tag{3}
\]

Then any weak-* limit \( f \) of \( f_\varepsilon \) is monokinetic, \( f = \rho(t, x) \delta(v - u(t, x)) \) for a.e. \( t \), where \( \rho, u \) are a solution in the sense of distributions to the pressureless system (2) while \( u \) has the Oleinik property for any \( t > 0 \)

\[
u(t, x) - u(t, y) \leq \frac{x - y}{t}, \quad \text{for } \rho \text{ a.e. } x \geq y.
\]
A global, dissipated functional?

→ **Setting.** Assume that \( f \in L^\infty(\mathbb{R}_+, M^1(\mathbb{R}^2)) \) is a positive, bounded measure with compact support in \((x, v) \in [-R, R]^2\), and solves

\[
\partial_t f + v \partial_x f = -\partial_{vv} m / \varepsilon, \tag{4}
\]

where \( m \) is an unknown positive measure in \( M^1(\mathbb{R}_+ \times \mathbb{R}^2) \).

→ **A “new” functional.** For any \( t \geq 0, \eta > 0, k \geq 1, \mu > 0 \), define

\[
\mathcal{L}_{\eta, \mu, k}(f)(t) := \int \frac{(v - w)^{k+2}}{(x - y + \eta)^k} \chi\mu(x - y) f(t, x, v) f(t, y, w) \, dv \, dw \, dx \, dy
\]

\[\chi\mu(r)\]

\[0\]

\[\mu\]

\[1\]

\[r\]

\[^8\text{See also Bony (1987), Cercignani (1992), Biriuk, Craig, Panferov (2006)}\]
A global, dissipated functional?

→ Setting. Assume that $f \in L^\infty(\mathbb{R}_+, M^1(\mathbb{R}^2))$ is a positive, bounded measure with compact support in $(x, v) \in [-R, R]^2$, and solves

$$\partial_t f + v \partial_x f = -\partial_{vv} m/\varepsilon,$$

where $m$ is an unknown positive measure in $M^1(\mathbb{R}_+ \times \mathbb{R}^2)$.

→ A “new” functional. For any $t \geq 0$, $\eta > 0$, $k \geq 1$, $\mu > 0$, define

$$\mathcal{L}_{\eta, \mu, k}(f)(t) := \int \frac{(v - w)^{k+2}}{(x - y + \eta)^k} \chi_{\mu}(x - y) f(t, x, v) f(t, y, w) \, dv \, dw \, dx \, dy$$

→ First observation. Let $(x_i(t), v_i(t))_{1 \leq i \leq N}$ for $N \in \mathbb{N}$ a solution to the sticky particles system and $f_N$ the associated empirical measure. Then

$$0 \leq \mathcal{L}_{\eta, \mu, k}(f_N)(T) \leq C_{N, k}.$$

See also Bony (1987), Cercignani (1992), Biriuk, Craig, Panferov (2006)
A More General Result

Theorem (P.-E. Jabin, TR, 2016)

Consider a sequence $f_\varepsilon \in L^\infty([0, T], M^1(\mathbb{R}^2))$ of solutions to (4) with mass 1 for a corresponding sequence of non negative measures $m_\varepsilon$. Assume that for any $k$

$$
\sup_{\varepsilon} \sup_{t \in [0, T]} \int_{\mathbb{R}^2} |v|^k f_\varepsilon(t, dx, dv) < \infty, \quad \sup_{\varepsilon} \sup_{t \in [0, T]} \int_{\mathbb{R}^2} |x|^2 f_\varepsilon(t, dx, dv) < \infty.
$$

Assume moreover that $f_\varepsilon$ satisfies a (technical) trace condition on $t$ and that

$$
\sup_{\varepsilon} \sup_{\eta, \mu} \mathcal{L}_{\eta, \mu, 0}(f_\varepsilon)(t = 0) < \infty.
$$

Then any weak-* limit $f$ of $f_\varepsilon$ solves the sticky particles dynamics in the sense that $\rho = \int_{\mathbb{R}} f(t, x, dv)$ and $j = \int_{\mathbb{R}} v f(t, x, dv) = \rho u$ are a distributional solution to the pressureless system (2) while $u$ has the Oleinik property for any $t > 0$

$$
u(t, x) - u(t, y) \leq \frac{x - y}{t}, \quad \text{for } \rho \text{ a.e. } x \geq y.$$
Continuity properties

Proposition

Let \( \mu > 0, \eta > 0, k \geq 1 \) and \( f_n \in L^\infty \left(0, T; M^1(\mathbb{R}^2)\right) \) solutions to (4) with bounded moments in \( v \) of order \( k + 3 \) such that \( f_n \rightharpoonup f \in L^\infty \left(0, T; M^1(\mathbb{R}^2)\right) \). Then \( \mathcal{L}_{\eta, \mu, k}(f) \) is BV in \( t \) and we have for any \( 0 < s < t \)

\[
\int_s^t \mathcal{L}_{\eta, \mu, k}(f)(r) \, dr = \lim \int_s^t \mathcal{L}_{\eta, \mu, k}(f_n)(r) \, dr,
\]

\[
\int_s^t \mathcal{L}_{\eta, 0^+, k}(f)(r) \, dr = \lim \inf \int_s^t \mathcal{L}_{\eta, 0^+, k}(f_n)(r) \, dr.
\]
Continuity properties

Proposition

Let $\mu > 0$, $\eta > 0$, $k \geq 1$ and $f_n \in L^\infty \left(0, T; M^1(\mathbb{R}^2)\right)$ solutions to (4) with bounded moments in $v$ of order $k + 3$ such that $f_n \rightharpoonup f \in L^\infty \left(0, T; M^1(\mathbb{R}^2)\right)$. Then $\mathcal{L}_{\eta,\mu,k}(f)$ is BV in $t$ and we have for any $0 < s < t$

\[
\int_s^t \mathcal{L}_{\eta,\mu,k}(f)(r) \, dr = \lim \int_s^t \mathcal{L}_{\eta,\mu,k}(f_n)(r) \, dr,
\]

\[
\int_s^t \mathcal{L}_{\eta,0+,k}(f)(r) \, dr = \lim \inf \int_s^t \mathcal{L}_{\eta,0+,k}(f_n)(r) \, dr.
\]

Main identity involved in the proof. Using the equation, one has

\[
\frac{d}{dt} \mathcal{L}_{\eta,\mu,k}(f)(t) = \int \left\{ (v - w)^{k+3} f f' \left[ -k \frac{\chi_{\mu}(x - y)}{(x - y + \eta)^{k+1}} + \frac{\chi_{\mu}'(x - y)}{(x - y + \eta)^k} \right] \\
- (k + 1) (k + 2) \left[ f' m + f m' \right] \frac{(v - w)^k}{(x - y + \eta)_+^k} \chi_{\mu}(x - y) \right\} dv \, dw \, dx \, dy.
\]
Theorem

Assume that \( f \in L^\infty([0, T], M^1(\mathbb{R}^2)) \) solves (4), and has bounded \( v \)-moments for some \( k \geq 0 \)

\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^2} |v|^{k+2} f(t, x, v) \, dx \, dv < \infty.
\]

Assume moreover that

\[
\sup_{\mu, \eta} \int_0^T \mathcal{L}_{\eta, \mu, k}(f)(r) \, dr < \infty.
\]

Then \( f \) is monokinetic for a.e. \( t \): There exist \( \rho \in L^\infty([0, T], M^1(\mathbb{R})) \), \( u \in L^\infty([0, T], L^{k+2}(\rho)) \) s.t. for a.e. \( t \)

\[
f(t, x, v) = \rho(t, x) \delta(v - u(t, x)).
\]
Connection with Monokinetic Solutions

**Theorem**

Assume that $f \in L^\infty([0, T], M^1(\mathbb{R}^2))$ solves (4), and has bounded $v$-moments for some $k \geq 0$

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^2} |v|^{k+2} f(t, x, v) \, dx \, dv < \infty.$$  

Assume moreover that

$$\sup_{\mu, \eta} \int_0^T \mathcal{L}_{\eta, \mu, k}(f)(r) \, dr < \infty.$$  

Then $f$ is monokinetic for a.e. $t$: There exist $\rho \in L^\infty([0, T], M^1(\mathbb{R}))$, $u \in L^\infty([0, T], L^{k+2}(\rho))$ s.t. for a.e. $t$

$$f(t, x, v) = \rho(t, x) \delta(v - u(t, x)).$$

**Some elements of proof.** By Radon-Nikodym Theorem, one has that $f(t, x, v) = \rho(t, x) M(t, x, v)$ and the goal is to show that $M$ is concentrated on Dirac deltas.
Elements of Proof

We proceed in two steps by considering the atomic and non-atomic parts of $\rho$:

$$\rho(t, x) = \sum_{n=1}^{\infty} \rho_n(t) \delta(x - x_n(t)) + \tilde{\rho}(t, x),$$

where $\tilde{\rho}$ does not contain any Dirac deltas.
**Elements of Proof**

We proceed in two steps by considering the atomic and non-atomic parts of $\rho$:

$$\rho(t, x) = \sum_{n=1}^{\infty} \rho_n(t) \delta(x - x_n(t)) + \tilde{\rho}(t, x),$$

where $\tilde{\rho}$ does not contain any Dirac deltas.

**Control of the non-atomic part.** Using only the functional $\mathcal{L}_{\eta, \mu, k}$ (and not the equation), we show that $\text{supp } M(t, x, \cdot) \subset (-\infty, u(t, x)] \cap [u(t, x), +\infty)$. But since for $\rho(t, dx) dt$-almost any $t$ and $x$,

$$\int_{\mathbb{R}} M(t, x, dv) = u(t, x),$$

hence $M(t, x, v) = \delta(v - u(t, x))$. 
Elements of Proof

We proceed in two steps by considering the atomic and non-atomic parts of $\rho$:

$$
\rho(t, x) = \sum_{n=1}^{\infty} \rho_n(t) \delta(x - x_n(t)) + \tilde{\rho}(t, x),
$$

where $\tilde{\rho}$ does not contain any Dirac deltas.

**Control of the non-atomic part.** Using only the functional $\mathcal{L}_{\eta, \mu, \kappa}$ (and not the equation), we show that $\text{supp } M(t, x, \cdot) \subset (-\infty, u(t, x)] \cap [u(t, x), +\infty)$. But since for $\rho(t, dx) dt$-almost any $t$ and $x$,

$$
\int_{\mathbb{R}} M(t, x, dv) = u(t, x),
$$

hence $M(t, x, v) = \delta(v - u(t, x))$.

**Control of the atomic part.** Given that $f$ is $BV$ in time, by contradiction if $f$ is not monokinetic at $a.e. \ t$ then there exists $t_0, x_0, \rho_0 > 0$ and $M_0(v) \neq \delta(v - u(t_0, x_0))$ s.t.

$$
f(t_0+, x, v) = g + \rho_0 \delta(x - x_0) M_0(v), \quad g \geq 0, \quad \int_{\mathbb{R}} M_0(dv) = 1.
$$

We use Eq. (4) to show that the atom at $x_0$ then has to split at $t > t_0+$. The corresponding pieces will now necessarily interact in $\mathcal{L}_{\eta, 0+, \kappa}(f)(t)$, not being at the same point and this will lead to a contradiction.
We have showed that in dimension 1, the pressureless Euler equation can be obtained as the hydrodynamic limit of an abstract equations which generalizes the granular gases equation;

We exhibited a new dissipated functional for $1d_x - 1d_v$ energy-dissipative kinetic equations, hence providing a new tool to obtain estimates;

Extension to higher dimensions seems unfortunately impossible because of the techniques considered (Cercignani functionals);

We believe that the general result can be applied to a lot of kinetic models, such as generalized viscoelastic hard spheres ones or kinetic description of flocking phenomena (Vicsek model).
We have showed that in dimension 1, the pressureless Euler equation can be obtained as the hydrodynamic limit of an abstract equations which generalizes the granular gases equation;

We exhibited a new dissipated functional for $1\,dx - 1\,dv$ energy-dissipative kinetic equations, hence providing a new tool to obtain estimates;

Extension to higher dimensions seems unfortunately impossible because of the techniques considered (Cercignani functionals);

We believe that the general result can be applied to a lot of kinetic models, such as generalized viscoelastic hard spheres ones or kinetic description of flocking phenomena (Vicsek model).

Thank you for your attention!