Hydrodynamic Limits of Granular Gases to Pressureless Euler in Dimension 1

Pierre-Emmanuel Jabin and Thomas Rey

Laboratoire Paul Painlevé
Université Lille 1 & Inria RAPSODI

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Outline of the Talk

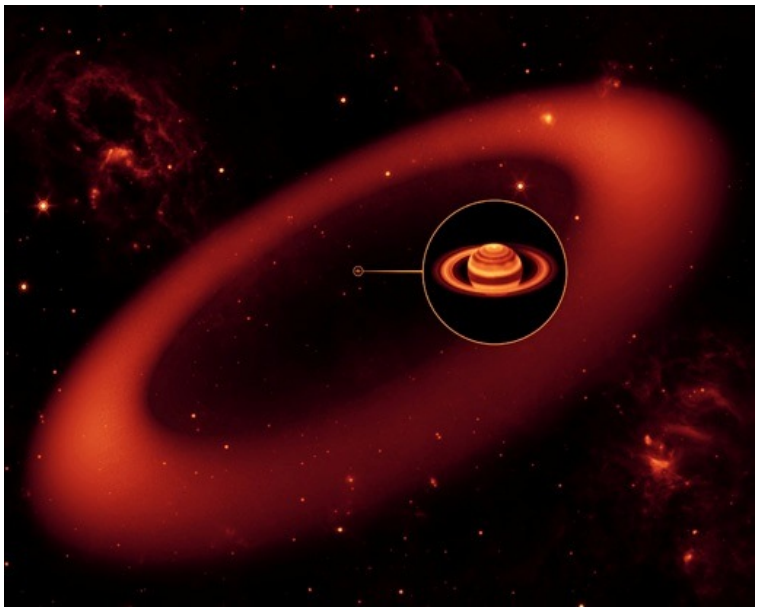
1. Generalities on the Granular Gases Equation
   - Introduction
   - Modeling
2. Macroscopic Descriptions
3. Presureless Euler Limit
   - Mathematical Results
   - The functional $\mathcal{L}_{\eta,\mu,k}$

The Granular Gases Equation

The granular gases equation describes the behavior of a dilute gas of particles when the only interactions taken into account are binary inelastic collisions

Applications

Pollen dissemination, avalanches, planetary rings, ...



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[1] Courtesy of T. Pöschel

The clustering phenomenon

Numerical method. Rescaling velocity method, using some qualitative properties of the equation[2].

Results. Local density, $\varepsilon = 0.05$: granular gas (left) vs. perfect elastic gas (right), at time $T = 2$.

Numerical parameters. $2d_x \times 3d_v$ model, $N_x = N_y = 100$, $N_{v_x} = N_{v_y} = N_{v_z} = 32$.

$\rightarrow$ Consistent with experiments (Brillantov, Pöschel, 2004).

[2] F. Filbet, TR, *JCP* (2013)

A Boltzmann-like Kinetic Equation

General Scaled Equation

Study of a particle distribution function $f^\varepsilon(t,x,v)$, depending on time $t>0$, space $x \in \Omega \subset \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$, solution to

$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + v \cdot \nabla_x f^\varepsilon = \dfrac{1}{\varepsilon} \mathcal{Q}_e(f^\varepsilon, f^\varepsilon), \\ \\ f^\varepsilon(0,x,v) = f_0(x,v), \end{cases} \tag{1}$$

where $\mathcal{Q}_e$ is the inelastic collision operator, describing the microscopic collision dynamic and ε is a scaling parameter.

- $\rightarrow$ ε is the Knudsen number, ratio of the mean free path between collision by the typical length scale of the problem;
- $\rightarrow$ $\mathcal{Q}_e$ only acts on the v variable;
- $\rightarrow$ Boundary conditions in space needed to describe completely the problem.

Modeling Assumptions

Microscopic dynamics

- Localized interactions: particles interact only by contact (collision), at a given time t and a given position x;
- Diluted gases: collisions occur between two particles at the same time (we neglect the collisions of three particles or more);
- ~~Micro-reversible collisions: the collision dynamics is time-reversible (at the microscopic level)~~;
- Boltzmann chaos assumption: the velocity of two colliding particles are uncorrelated before collision.

The microscopic collision process is said to be

$\rightarrow$ elastic when the kinetic energy is conserved during a collision (this is for example the case for a perfect molecular gas);

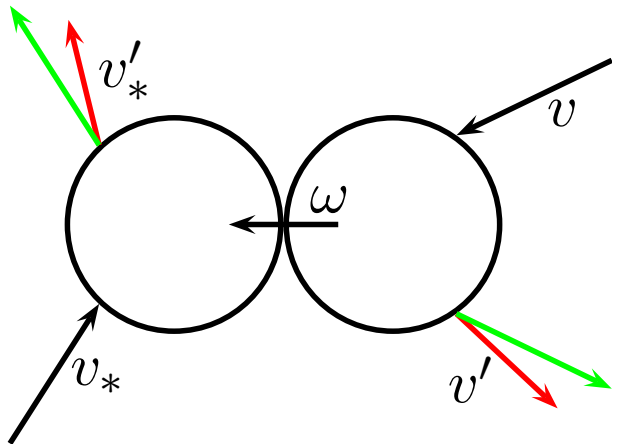
$\rightarrow$ inelastic when a fraction of the kinetic energy is dissipated during a collision (this is the case of a granular gas).

Inelastic Hard Spheres

- Microscopic dynamics: conservation of impulsion and dissipation of kinetic energy;
- Constant restitution coefficient $e \in [0,1]$;
- Parametrization of the post-collisional velocities (v', v'_*) as a function of the pre-collisional velocities (v, v_*):

$$\begin{cases} v' = v - \dfrac{1+e}{2}\left((v-v_*)\cdot\omega\right)\omega, \\ \\ v'_* = v_* + \dfrac{1+e}{2}\left((v-v_*)\cdot\omega\right)\omega, \end{cases}$$

where $\omega \in \mathbb{S}^{d-1}$ is the impact direction;



(green is elastic, red is inelastic)

Features of the Model

If $e < 1$ (true inelasticity),

- Trivial equilibria (Dirac masses) because of the temperature dissipation: linearized study difficult;
- Particle velocities highly correlated: non Gaussian tails;
- Mathematical analysis possible only in the L^1 Banach setting: no scalar products, perturbative theory intricate;
- No formal entropy dissipation structure[3]: very few energy estimates available, large time behavior difficult to investigate;
- Macroscopic description formally given by the pressureless gas dynamics: creation of singularities in finite time;
- Creation of spatial d'inhomogeneities and zero temperature zones (clustering phenomenon).
- Meaningful even in dimension 1: the collision process is given by

$$\{v', v'_*\} = \{v, v_*\} \quad \text{or} \quad \left\{ \frac{v + v_*}{2} \pm \frac{e}{2}(v - v_*) \right\}.$$

[3] Nevertheless, extensive numerical experiments show that the Boltzmann entropy would be a good candidate, see Garcia, Maynar, Mischler, Mouhot, TR, Trizac, *JSM* (2015)

The Inelastic Collision Operator

For two velocities $v, v_* \in \mathbb{R}^d$, we set $u := v - v_*$, $\hat{u} := \frac{u}{|u|}$; if ψ is a smooth test function, $\psi' := \psi(v')$, $\psi_* := \psi(v_*)$, $\psi'_* := \psi(v'_*)$.

Representations of the collision operator

- Weak form of the collision operator:

$$\int_{\mathbb{R}^d} \mathcal{Q}_e(f,f)(v)\,\psi(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} |u|\, f_* f\ (\psi' + \psi'_* - \psi - \psi_*)\, b(\hat{u}\cdot\sigma)\,d\sigma\,dv\,dv_*,$$

  where $b \geq 0$ the collisional cross section.

- Strong form: if e is non-zero,

$$\mathcal{Q}_e(f,f)(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |u| \left(\frac{1}{e^2}\,{}'f\,{}'f_* - f\,f_*\right) b(\hat{u}\cdot\sigma)\,d\sigma\,dv_*.$$

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Macroscopic Properties of the Operator

Conservation of mass and momentum, dissipation of kinetic energy:

$$\int_{\mathbb{R}^d} \mathcal{Q}_e(f,f)(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \\ -(1-e^2)D(f,f) \end{pmatrix}$$

where $D(f,f) \geq 0$ is the energy dissipation functional:

$$\begin{aligned} D(f,f) &:= b_1 \int_{\mathbb{R}^d \times \mathbb{R}^d} f\, f_*\, |v - v_*|^3 \, dv\, dv_*, \\ &\geq b_1 \rho^{5/2} \left(\int_{\mathbb{R}} f(v)\, |v-u|^2 dv \right)^{\frac{3}{2}} \geq 0. \end{aligned}$$

and b_1 is the angular momentum:

$$b_1 := \int_{\mathbb{S}^{d-1}} (1 - (\widehat{u} \cdot \omega))\, b(\widehat{u} \cdot \omega)\, d\omega < \infty.$$

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Microscopic consequence: Concentration in the velocity space!

Monokinetic Distribution and Presureless Dynamics

Define the first moments of f as

$$(\rho, \rho \boldsymbol{u}, \mathcal{E}) = \int_{\mathbb{R}^d} f(v)\varphi(v)\,dv, \qquad \varphi(v) = (1, v, |v|^2/2).$$

Dissipation of energy formally yields that when $\varepsilon \to 0$,

$$f(t,x,v) \rightharpoonup_{\varepsilon \to 0} \rho(x)\delta_0\left(v - \boldsymbol{u}(x)\right), \quad \forall (x,v) \in \Omega \times \mathbb{R}^d,$$

where, using the macroscopic properties of $\mathcal{Q}_e$,

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \boldsymbol{u}) = 0, \\ \\ \partial_t (\rho\, \boldsymbol{u}) + \nabla_x \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) = \mathbf{0}. \end{cases}$$

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Presureless Euler/Sticky particles equation: Generation of δ-singularity in finite time and transient clusters formation.

Quasi-Elastic Limit and Compressible Euler Dynamics

One can show[4] that if $1-e \sim \varepsilon$, then

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{\mathcal{Q}_1(f,f)}{\varepsilon} + \frac{1-e}{\varepsilon}\mathcal{I}(f,f) + \mathcal{O}\left(\frac{(1-e)^2}{\varepsilon}\right).$$

for an explicit friction operator $\mathcal{I}$.
When $\varepsilon \to 0$, there is a compressible Euler-like dynamics[5]

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \boldsymbol{u}) = 0, \\ \partial_t(\rho\, \boldsymbol{u}) + \nabla_x \cdot (\rho(\boldsymbol{u} \otimes \boldsymbol{u} + T\boldsymbol{I})) = \boldsymbol{0}, \\ \partial_t \mathcal{E} + \nabla_x \cdot (\boldsymbol{u}(\mathcal{E} + \rho T)) = -C_d \rho^2 T^{\frac{3}{2}}, \end{cases}$$

which also exhibits cluster formation[6]!

---

[4] G. Toscani (2004)

[5] The closure is made thanks to the Maxwellian

$$\mathcal{M}_{\rho,\boldsymbol{u},T}(v) = \frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{|v-\boldsymbol{u}|^2}{2T}\right).$$

[6] Carrillo, Salueña (2006)

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A State of the Art on the Granular Gases Equation

Space homogeneous setting $\partial_t f = \mathcal{Q}_e(f,f)$

- Cauchy problem for different types of kernels: Bobylev-Carrillo-Gamba (2000), Bobylev-Cercignani-Toscani (2003), Mischler-Mouhot-Ricard (2006);
- Qualitative behavior: Bobylev-Gamba-Panferov (2004), Gamba-Panferov-Villani (2004);
- Cooling process and asymptotic/self-similar behavior: Li-Toscani (2004), Mischler-Mouhot (2006), Alonso-Lods (2010, 2012), TR (2012);
- Stability and convergence towards the (unique) self-similar solution: Mischler-Mouhot (2009), Alonso-Lods (2011, 2013).

Full space-dependent equation $\partial_t f + v \cdot \nabla_x f = \mathcal{Q}_e(f,f)$

- Cauchy problem 1d: Benedetto-Caglioti-Pulvirenti (1997, 2001);
- Cauchy problem near vacuum: Alonso (2009);
- Cauchy problem quasi-homogeneous, in the torus, thermal bath: Tristani (2013);
- Hydrodynamic limits: Toscani (2004), Carlen-Chow-Grigo (2010), TR (2014), Jabin-TR (2016).

A State of the Art on the Pressureless Euler Equation

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \boldsymbol{u}) = 0, \\ \\ \partial_t (\rho\, \boldsymbol{u}) + \nabla_x \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) = \mathbf{0}. \end{cases}$$

Attract a lot of interest from the physics community because of its description of the large scale structure of the universe: Silk-Szalay-Zeldovich (1983)

- A rather delicate equation. Exhibits shocks because u (formally) solves Burgers equation where $\rho > 0 \Rightarrow$ concentration in $\rho \Rightarrow$ ill-posed system!
- Nevertheless, wellposedness theory exists by imposing semi-Lipschitz condition on u: Bouchut-James (1999), Boudin (2000), Huang-Wang (2001-2004);
- Model obtained as the hydrodynamic limit of the sticky particle model
  - In dimension 1: E-Rykov-Sinai (1996), Brenier-Grenier (1998),
  - In dimension 2: Berthon-Breuss-Titeux (2006), Chertock-Kurganov-Rykov (2007),

One Dimensional Sticky Particles

→ Known results. Hydrodynamic limit of the one dimensional N–particles system with sticky collisions[7] towards the presureless Euler equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0, \\ \partial_t(\rho\, u) + \partial_x\left(\rho\, u^2\right) = 0. \end{cases} \tag{2}$$

→ Method. Convergence of the empirical density towards a monokinetic distribution, using (among other arguments) the (microscopic) Oleinik property

$$\sup_{i\in\{1,\dots,N\}} \frac{(v_{i+1}-v_i)_+}{(x_{i+1}-x_i)_+} < \frac{1}{t},$$

where (x_i, v_i) represents the position and velocity of the i^{th} particle at time t.

[7] after collisions, the two particles stick together and travel with their average velocity.

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where (x_i, v_i) represents the position and velocity of the i^{th} particle at time t.

Goal.

Convergence of the particle distribution function $f(t,x,v)$ solution to the granular gases equation towards the monokinetic density

$$f(t,x,v) = \rho(t,x)\,\delta\left(x - u(t,x)\right),$$

where $(\rho,\ u)$ solution to the presureless Euler system (2).

[7] after collisions, the two particles stick together and travel with their average velocity.

Hydrodynamic Limit of the Granular Gases Equation

Theorem (P.-E. Jabin, TR, 2016)

Consider a sequence of weak solutions $f_\varepsilon(t,x,v) \in L^\infty([0,\ T],\ L^p(\mathbb{R}^2))$ *for some* $p>2$ *and with total mass* 1 *to the granular gases Eq.* (1) *such that for any* k

$$\sup_\varepsilon \int_{\mathbb{R}^2} |v|^k f_\varepsilon^0(x,v)\,dx\,dv < \infty, \qquad \sup_\varepsilon \int_{\mathbb{R}^2} |x|^2 f_\varepsilon^0(x,v)\,dx\,dv < \infty,$$

and f_ε^0 *is, uniformly in* ε*, in some* L^p *for* $p>1$

$$\sup_\varepsilon \int_{\mathbb{R}^2} (f_\varepsilon^0(x,v))^p\,dx\,dv < \infty. \tag{3}$$

Then any weak- limit* f *of* f_ε *is monokinetic,* $f=\rho(t,x)\,\delta(v-u(t,x))$ *for* $a.e.\ t$*, where* $\rho,\ u$ *are a solution in the sense of distributions to the pressureless system* (2) *while* u *has the Oleinik property for any* $t>0$

$$u(t,x)-u(t,y) \le \frac{x-y}{t}, \quad \text{for } \rho\ a.e.\ x \ge y.$$

A global, dissipated functional?

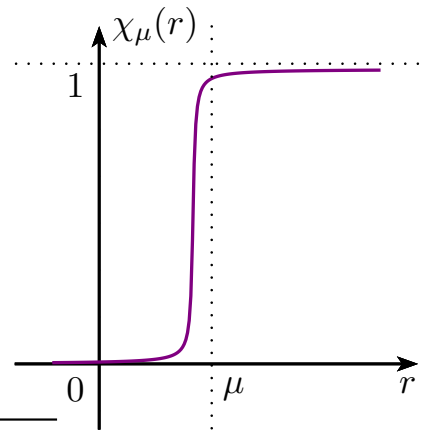
→ Setting. Assume that $f \in L^\infty(\mathbb{R}_+,\ M^1(\mathbb{R}^2))$ is a positive, bounded measure with compact support in $(x,v) \in [-R,\ R]^2$, and solves

$$\partial_t f + v\,\partial_x f = -\partial_{vv} m/\varepsilon, \tag{4}$$

where m is an unknown positive measure in $M^1(\mathbb{R}_+ \times \mathbb{R}^2)$.

→ A "new" functional.[8] For any $t \geq 0$, $\eta > 0$, $k \geq 1$, $\mu > 0$, define

$$\mathcal{L}_{\eta,\mu,k}(f)(t) := \int \frac{(v-w)_+^{k+2}}{(x-y+\eta)^k} \chi_\mu(x-y)\, f(t,x,v)\, f(t,y,w)\, dv\, dw\, dx\, dy$$



[8] See also Bony (1987), Cercignani (1992), Biriuk, Craig, Panferov (2006)

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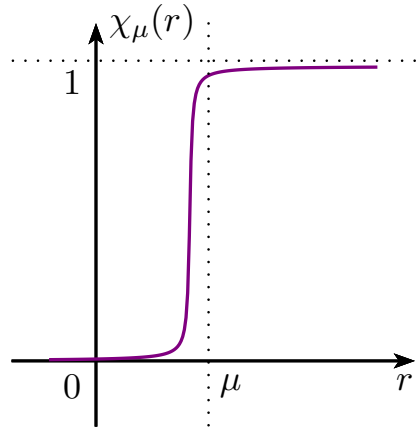
→ First observation. Let $(x_i(t), v_i(t))_{1\leq i\leq N}$ for $N \in \mathbb{N}$ a solution to the sticky particles system and f_N the associated empirical measure. Then

$$0 \leq \mathcal{L}_{\eta,\mu,k}(f_N)(T) \leq C_{N,k}.$$



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A More General Result

Theorem (P.-E. Jabin, TR, 2016)

Consider a sequence $f_\varepsilon \in L^\infty([0,\ T],\ M^1(\mathbb{R}^2))$ *of solutions to* (4) *with mass* 1 *for a corresponding sequence of non negative measures* m_ε. *Assume that for any* k

$$\sup_\varepsilon \sup_{t\in[0,\ T]} \int_{\mathbb{R}^2} |v|^k f_\varepsilon(t, dx, dv) < \infty, \quad \sup_\varepsilon \sup_{t\in[0,\ T]} \int_{\mathbb{R}^2} |x|^2 f_\varepsilon(t, dx, dv) < \infty.$$

Assume moreover that f_ε *satisfies a (technical) trace condition on* t *and that*

$$\sup_\varepsilon \sup_{\eta,\mu} \mathcal{L}_{\eta,\mu,0}(f_\varepsilon)(t=0) < \infty.$$

Then any weak- limit* f *of* f_ε *solves the sticky particles dynamics in the sense that* $\rho = \int_{\mathbb{R}} f(t,x,dv)$ *and* $j = \int_{\mathbb{R}} v\, f(t,x,dv) = \rho\, u$ *are a distributional solution to the pressureless system* (2) *while* u *has the Oleinik property for any* $t > 0$

$$u(t,x) - u(t,y) \leq \frac{x-y}{t}, \quad \text{for } \rho \text{ a.e. } x \geq y.$$

Continuity properties

Proposition

Let $\mu > 0$, $\eta > 0$, $k \geq 1$ *and* $f_n \in L^\infty\left(0, T; M^1(\mathbb{R}^2)\right)$ *solutions to* (4) *with bounded moments in* v *of order* $k+3$ *such that* $f_n \rightharpoonup f \in L^\infty\left(0, T; M^1(\mathbb{R}^2)\right)$. *Then* $\mathcal{L}_{\eta,\mu,k}(f)$ *is BV in* t *and we have for any* $0 < s < t$

$$\int_s^t \mathcal{L}_{\eta,\mu,k}(f)(r)\, dr = \lim \int_s^t \mathcal{L}_{\eta,\mu,k}(f_n)(r)\, dr,$$

$$\int_s^t \mathcal{L}_{\eta,0+,k}(f)(r)\, dr = \liminf \int_s^t \mathcal{L}_{\eta,0+,k}(f_n)(r)\, dr.$$

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$$\int_s^t \mathcal{L}_{\eta,0+,k}(f)(r)\, dr = \liminf \int_s^t \mathcal{L}_{\eta,0+,k}(f_n)(r)\, dr.$$

Main identity involved in the proof. Using the equation, one has

$$\frac{d}{dt}\mathcal{L}_{\eta,\mu,k}(f)(t) = \int \left\{ (v-w)_+^{k+3} f f' \left[-k \frac{\chi_\mu(x-y)}{(x-y+\eta)^{k+1}} + \frac{\chi'_\mu(x-y)}{(x-y+\eta)^k} \right] \right.$$
$$\left. - (k+1)(k+2) \left[f' m + f m'\right] \frac{(v-w)_+^k}{(x-y+\eta)^k} \chi_\mu(x-y) \right\} dv\, dw\, dx\, dy.$$

Connection with Monokinetic Solutions

Theorem

Assume that $f \in L^\infty([0,\ T],\ M^1(\mathbb{R}^2))$ *solves* (4), *and has bounded* v*-moments for some* $k \geq 0$

$$\sup_{t\in[0,\ T]} \int_{\mathbb{R}^2} |v|^{k+2} f(t,x,v)\, dx\, dv < \infty.$$

Assume moreover that

$$\sup_{\mu,\eta} \int_0^T \mathcal{L}_{\eta,\mu,k}(f)(r)\, dr < \infty.$$

Then f *is monokinetic for* $a.e.\ t$*: There exist* $\rho \in L^\infty([0,\ T],\ M^1(\mathbb{R}))$, $u \in L^\infty([0,\ T],\ L^{k+2}(\rho))$ *s.t. for* $a.e.\ t$

$$f(t,x,v) = \rho(t,x)\, \delta(v - u(t,x)).$$

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$$f(t,x,v) = \rho(t,x)\, \delta(v - u(t,x)).$$

Some elements of proof. By Radon-Nikodym Theorem, one has that $f(t,x,v) = \rho(t,x)\, M(t,x,v)$ and the goal is to show that M is concentrated on Dirac deltas.

Elements of Proof

We proceed in two steps by considering the atomic and non-atomic parts of ρ:

$$\rho(t,x) = \sum_{n=1}^{\infty} \rho_n(t)\,\delta(x - x_n(t)) + \tilde{\rho}(t,x),$$

where $\tilde{\rho}$ does not contain any Dirac deltas.

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where $\tilde{\rho}$ does not contain any Dirac deltas.

Control of the non-atomic part. Using only the functional $\mathcal{L}_{\eta,\mu,k}$ (and not the equation), we show that $\operatorname{supp} M(t,x,\cdot) \subset (-\infty, u(t,x)] \cap [u(t,x), +\infty)$. But since for $\rho(t,dx)\,dt$-almost any t and x,

$$\int_{\mathbb{R}} M(t,x,dv) = u(t,x),$$

hence $M(t,x,v) = \delta(v - u(t,x))$.

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$$\int_{\mathbb{R}} M(t,x,dv) = u(t,x),$$

hence $M(t,x,v) = \delta(v - u(t,x))$.

Control of the atomic part. Given that f is BV in time, by contradiction if f is not monokinetic at $a.e.\ t$ then there exists t_0, x_0, $\rho_0 > 0$ and $M_0(v) \neq \delta(v - u(t_0,x_0))$ s.t.

$$f(t_0+,x,v) = g + \rho_0\,\delta(x - x_0)\,M_0(v), \quad g \geq 0, \quad \int_{\mathbb{R}} M_0(dv) = 1.$$

We use Eq. (4) to show that the atom at x_0 then has to split at $t > t_0+$. The corresponding pieces will now necessarily interact in $\mathcal{L}_{\eta,0+,k}(f)(t)$, not being at the same point and this will lead to a contradiction.

- We have showed that in dimension 1, the pressureless Euler equation can be obtained as the hydrodynamic limit of an abstract equations which generalizes the granular gases equation;
- We exhibited a new dissipated functional for $1d_x - 1d_v$ energy-dissipative kinetic equations, hence providing a new tool to obtain estimates;
- Extension to higher dimensions seems unfortunately impossible because of the techniques considered (Cercignani functionals);
- We believe that the general result can be applied to a lot of kinetic models, such as generalized viscoelastic hard spheres ones or kinetic description of flocking phenomena (Vicsek model).

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Thank you for your attention!