Variations of Kac's model of a particle system

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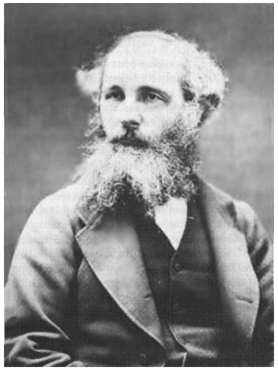
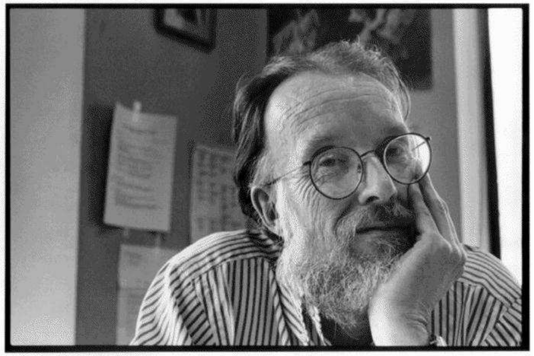
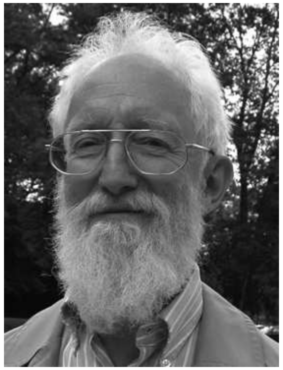
Plan of the talk

- Particle systems and kinetic equations
- Kac model of a particle system and propagation of chaos
- A relativistic version of the Kac model
- A particle system for the BGK equation



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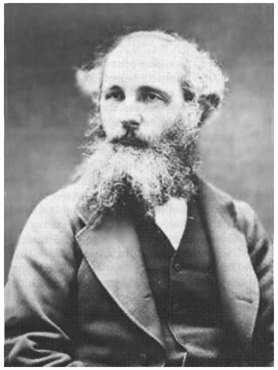


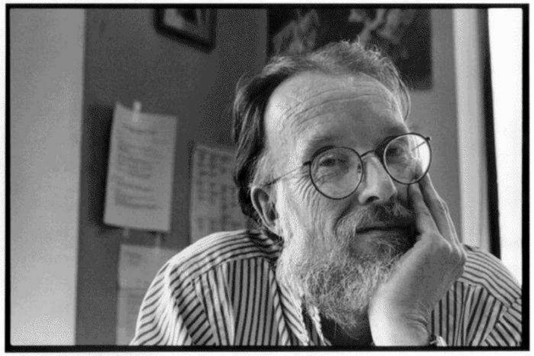
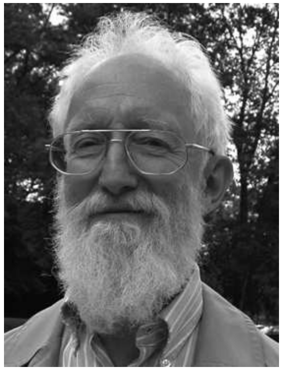

- Based on Dawan Mustafa's thesis *Propagation of chaos for Kac-like particle models*, dec 2015.

Particle systems

- Particles

  $z = (z_1, ..., z_N)$ $\qquad\qquad$ $z_j \in Z^N$

- Evolution (deterministic or stochastic)

  $z(t) \; = \; T_N^t z(0)$

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- \- Deterministic: Liouville equation
  \- Stochastic: Master equation (Kolmogorov equation)

Particle systems

- Marginals:

$$F_N^k(z_1, ..., z_k) = \int F_N(z_1, ..., z_k, z_{k+1}, ..., z_N)\, dz_{k+1} \cdots dz_N$$

$$f(t,z) \equiv \lim_{N\to\infty} F_N^1(t,z)$$

- Goals:
  - to prove that the limit exists
  - to prove that $f(t,z)$ satsifies an equation

$$\frac{\partial}{\partial t} f(t,v) = G[f](t,v)$$

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- Key concepts:
  - Symmetry: the densities $F_N(v_1,\ldots,v_N)$ invariant under permutation of indices
  - Chaos and propagation of chaos

Propagation of chaos according to Kac

Definition: A sequence of probabiliy measures $F_N(z_1, ..., z_N)$, $N = 1, ..., \infty$ is said to have the Boltzmann property, or to be chaotic if for each k,

$$\lim_{N\to\infty} F_N^k(z_1, ..., z_k) \to \prod_{j=1}^{k} \lim_{N\to\infty} F_N^1(z_j, t)$$

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Definition: Propagation of chaos is said to hold if whenever the initial data is chaotic, then so is the distribution at later times.

Kac's microscopic model

- N-particle system:

  $V = (v_1, ..., v_N)$, $v_i \in \mathbb{R}$ (the *master vector*)

- Phase space:

$$S^{N-1}(\sqrt{N})) = \left\{ (v_1, ..., v_N) \,\middle|\, v_1^2 + ... + v_N^2 = N \right\}$$

- Markov jump process:

- Exponentially distributed intervals between jumps:

$$\Delta t \sim e^{-Nt}$$

- Jumps involve two coordinates:

$$(v_1, ..., v_i, ..., v_j, ...v_N) \mapsto (v_1, ..., v_i', ..., v_j', ...v_N)$$

Kac's microscopic model

- Jumps
  - i Choose a pair (i, j) uniformly from $1 \leq i < j \leq N$.
  - ii Choose $\theta \in [-\pi, \pi]$ with law μ
    (i.e., $P(\theta \in J) = \int_J \mu(d\theta)$)
  - iii let $(v_i', v_j') = (v_i \cos(\theta) - v_j \sin(\theta), v_i \sin(\theta) + v_j \cos(\theta))$

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- Notation: $V \mapsto V' = R_{i,j}(\theta)V$
- Energy conservation:
$$v_i'^2 + v_j'^2 = v_i^2 + v_j^2$$
- No conserved momentum:
$$v_i' + v_j' \neq v_i + v_j$$
  (in general)

Kac's master equation

- If $\mu(d\theta) = \dfrac{d\theta}{2\pi}$, then

$$\partial_t F_N(V,t) = \frac{2}{N-1} \sum_{1\le i<j\le N} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(F_N\left(R_{ij}(\theta)V,t\right) - F_N(V,t)\right)\, d\theta$$

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$$= L_N F_N(V,t) = N\frac{2}{N(N-1)} \sum_{1\le i<j\le N} L_{(i,j)} F_N(V,t)$$

Here L_N is a bounded, self adjoint operator

The BGK- equation

- $$\frac{\partial}{\partial t} f + v \cdot \nabla f = \frac{1}{\tau} \left(\Phi_{T,u,\rho} - f \right)$$

  $$\Phi_{T,u,\rho}(v) = \frac{\rho}{(2\pi T)^{-d/2}} \exp\left(-(v-u)^2/2T \right)$$

  $$f = f(x,v,t), \qquad T = T(x,t), \qquad u = u(x,t) \in \mathbb{R}^d, \qquad \rho = \rho(x,t)$$

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  $\Phi(v)$ standard Gaussian

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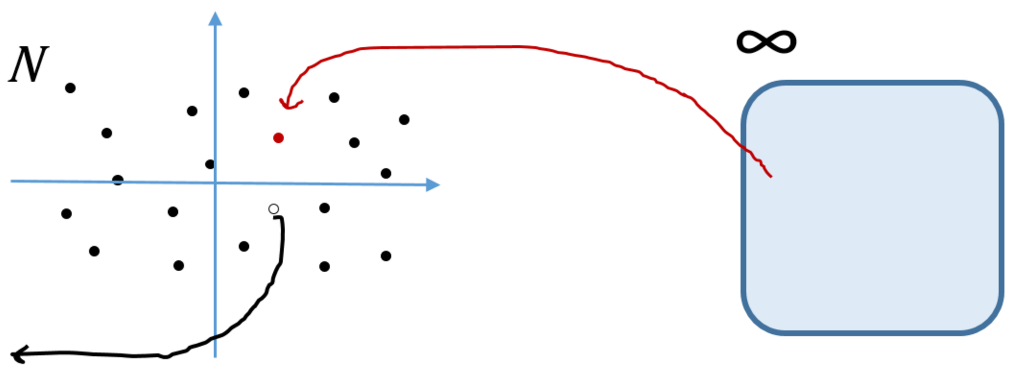
- 

- $(v_1, \ldots, v_j, \ldots, v_N) \mapsto (v_1, \ldots., \xi_j, \ldots., v_N) \qquad \mathrm{Law}_{\xi_j} = \Phi$

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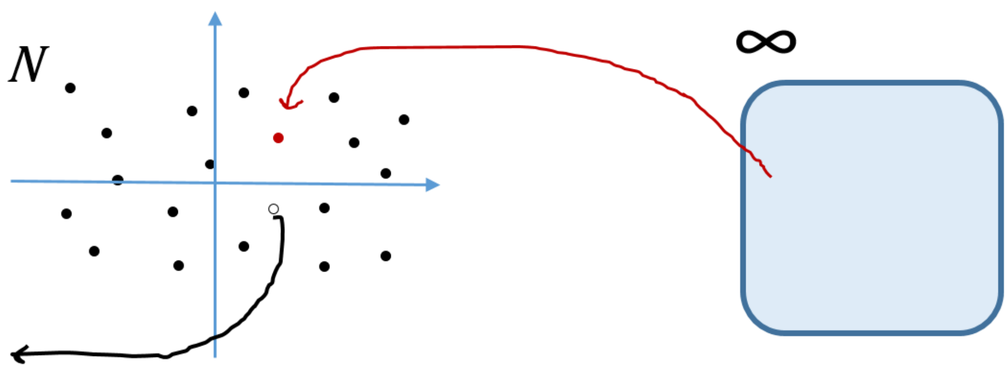
- 

- $(v_1, ..., v_j, ..., v_N) \mapsto (v_1, ..., \xi_j, ..., v_N) \qquad \mathrm{Law}_{\xi_j} = \Phi$

- $\frac{\partial}{\partial t} F_N(t, v_1, ..., v_N) =$

  $N \frac{1}{N} \sum_{j=1}^{N} \left(\int_{\mathbb{R}} F_N(t, v_1, ..., x_j, ..., v_N) dx_j \Phi(v_j) - F_N(t, v_1, ..., v_N) \right)$

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- The equation for the one-particle marginal:
  $$\frac{\partial}{\partial t} F_N^1(t, v_1) \; = \int_{\mathbb{R}} F_N^1(t, x_1)\, dx_1 \Phi(v_1) - F_N^1(t, v_1)$$

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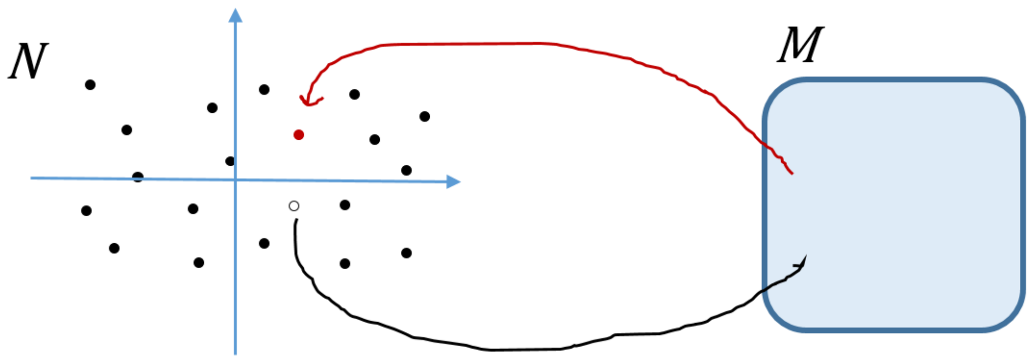
- 

$$T(v_1, \dots, v_N) = \frac{N+M-\sum v_j^2}{M}$$

- $(v_1, \dots, v_j, \dots, v_N) \mapsto (v_1, \dots., \sqrt{T(V)}\xi_j, \dots., v_N)$ $\qquad \mathrm{Law}_{\xi_j} = \Phi$

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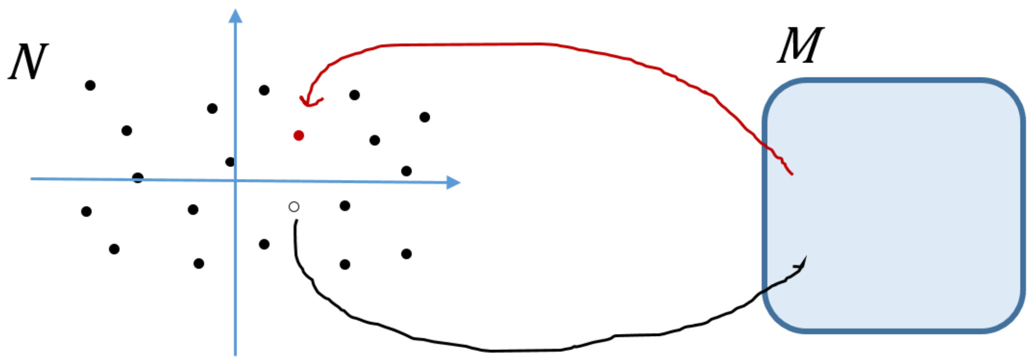
- 

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- $$\frac{\partial}{\partial t} F_N(t, v_1, ..., v_N) = \sum_{j=1}^{N} \left(\int_{\mathbb{R}} F_N(t, v_1, ..., x_j, ..., v_N) T(V')^{-1/2} \Phi\left(T(V')^{-1/2} v_j\right) dx_j - F_N(t, v_1, ..., v_N) \right)$$

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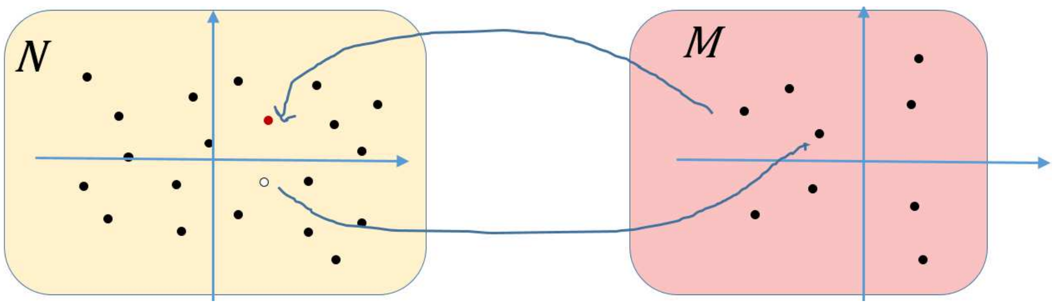
$$- F_N^1(t, v_1)$$

- Integrand depends only on $x_1^2 + v_2^2 + \cdots + v_N^2$
- $\mathbb{E}(T) \to 1$ when $N \to \infty$
- In fact: $T \to 1$ n.s.

The BGK equation

- 

- N passive particles $V = (v_1, ..., v_\mathrm{N})$
- M active particles $W = (w_1, ..., w_\mathrm{M})$
- Exchange between active and passive as before
- Pairwise collisions between active particles at very high rate
- $\mathrm{M}, \mathrm{N} \to \infty$
- $\mathrm{M}/\mathrm{N} \to 0$ when $\mathrm{N} \to \infty$

- Purpose: to get the BGK-equation from pure, conservative, particle system.

The BGK equation

- $F = F(V, W) = F(v_1, ..., v_N, w_1, ..., w_M) \in L^1(S^{N+M-1}(\sqrt{N+M}))$
- The exchange operator:

$$U_{NM\lambda_1} F(v_1, ..., v_N, w_1, ..., w_M) = \frac{\lambda_1}{M} \sum_{j=1}^{N} \sum_{k=1}^{M} \left(F(V^j_{w_k}, W^k_{v_j}) - F(V, W) \right)$$

- Collisions for active particles:

$$L_{NM\lambda_2} F(V, W) = \frac{2N\lambda_2}{M-1} \sum_{1 \leq j < k \leq M} \int_{-\pi}^{\pi} (F(V, R_{jk}(\theta) W) - F(V, W)) \, d\theta \,,$$

The BGK equation

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- The master equation:

$$\frac{\partial}{\partial t} F(V, W, t) = \left(L_{NM\lambda_2} + U_{NM\lambda_1} \right) F(V, W, t)$$

Marginals

- $V = (v_1, ..., v_n, v_{n+1}, ..., v_N) = (V_n, V^n)$
  $W = (w_1, ..., w_m, w_{m+1}, ..., w_M) = (W_m, W^m)$

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- Marginal

$$\int_{S^{N+M-1}(\sqrt{N+M})} \phi(V_n, W_m) F(V, W) d\sigma(V, W)$$

$$= \int_{\mathbb{R}^{n+m}} \phi(V_n, W_m) f^{(n,m)}_{N,M}(V_n, W_m)\, dV_n\, dW_m\,,$$

- Average

$$\overline{F}(V) = \int_{\Omega_{N,0}} F(V, W)\, d\sigma(W)\,.$$

A Theorem

- Assume $F(V,W,t)$ solves the master equation

$$\int_{\Omega_{0,0}} |F(V,W,0)|^2 \, d\sigma(V,W) \leq C_{M,N},$$

$$\int_{\Omega_{0,0}} F(V,W,0) v_1^4 \, d\sigma(V,W) \leq \infty.$$

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- Choose $M = M(N)$, $\lambda_2 = \lambda_2(N)$ with
  $N/M \to \infty$, $N/M^2 \to 0$ and
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  $N/M \to \infty$, $N/M^2 \to 0$ and
  $\lambda_2/C_{NM} \to \infty$ when $N \to \infty$.

- Then, for $0 < t_0 \le t \le T$, $\quad T < \infty$

  $$\lim_{N\to\infty} \int_{B_N} \frac{\partial}{\partial t} f_{N,M}^{N,0}(V,t) g(v_1)\, dV = \int_{\mathbb{R}} (\Phi(v_1) - f(v_1,t)) g(v_1)\, dv_1,$$

  where

  $$f(v_1,t) = \lim_{N\to\infty} f_{N,M}^{(1,0)}(v_1,t).$$

Elements of the proof

-

$$\frac{\partial}{\partial t} f_{N,M}^{(N,0)}(V,t) = \lambda_1 \sum_{j=1}^{N} \Gamma_{N,0} \int_{\Omega_{N,0}} \left(\bar{F}(V_{w_1}^{j},t) - \bar{F}(V,t)\right) \, d\sigma(W)$$

$$+ \lambda_1 \sum_{j=1}^{N} \Gamma_{N,0} \int_{\Omega_{N,0}} \left(F(V_{w_1}^{j}, W_{v_j}^{1}, t) - \bar{F}(V_{w_1}^{j},t)\right) \, d\sigma(W)$$

Elements of the proof

- $$\frac{\partial}{\partial t} f^{(N,0)}_{N,M}(V,t) = \lambda_1 \sum_{j=1}^{N} \Gamma_{N,0} \int_{\Omega_{N,0}} \left(\bar{F}(V^j_{w_1}, t) - \bar{F}(V,t) \right) \, d\sigma(W)$$
$$+ \lambda_1 \sum_{j=1}^{N} \Gamma_{N,0} \int_{\Omega_{N,0}} \left(F(V^j_{w_1}, W^1_{v_j}, t) - \bar{F}(V^j_{w_1}, t) \right) \, d\sigma(W)$$

- The first term:

$$\lambda_1 \sum_{j=1}^{N} \left(\int_{-\sqrt{\frac{N+M-|V|^2}{M}}}^{-\sqrt{\frac{N+M-|V|^2}{M}}} \frac{1}{\sqrt{\tau(V^j_{w_1})}} \Phi_M \left(\frac{v_j}{\sqrt{\tau(V^j_{w_1})}} \right) f^{(N,0)}_{N,M}(V^j_{w_1}, t) \, dw_1 \right.$$
$$\left. - f^{(N,0)}_{N,M}(V,t) \right)$$

Law of first component of a point on $|W|^2 = M\tau$.

Elements of the proof

- $$\frac{\partial}{\partial t} f_{N,M}^{(N,0)}(V,t) = \lambda_1 \sum_{j=1}^{N} \Gamma_{N,0} \int_{\Omega_{N,0}} \left(\overline{F}(V_{w_1}^j, t) - \overline{F}(V,t) \right) \, d\sigma(W)$$
$$+ \lambda_1 \sum_{j=1}^{N} \Gamma_{N,0} \int_{\Omega_{N,0}} \left(F(V_{w_1}^j, W_{v_j}^1, t) - \overline{F}(V_{w_1}^j, t) \right) \, d\sigma(W)$$

- second term: Use

$$F(V,W,t) = e^{tL_{NM\lambda_2}} F(V,W,0) + \int_0^t e^{(t-s)L_{NM\lambda_2}} U_{NM\lambda_1} F(V,W,s) \, ds.$$

Elements of the proof

- second term: Use

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-

$$\begin{aligned} \|F(V,\cdot,t) - \overline{F}(V,t)\|_2 &\leq e^{-\frac{tN\lambda_2}{2}} \|F(V,\cdot,0) - \overline{F}(V,0)\|_2 \\ &+ \frac{1-e^{-\frac{tN\lambda_2}{2}}}{2N\lambda_2} \sup_{0\leq s\leq t} \|U_{NM}F(V,\cdot,s) - \overline{U_{NM}F}(V,s)\|_2, \end{aligned}$$

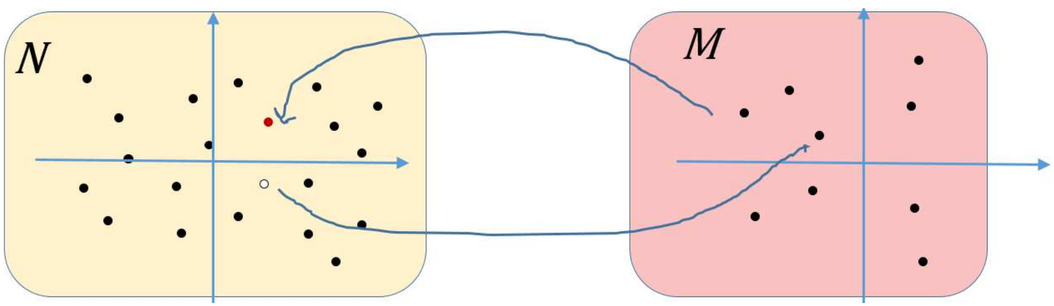
Elements of the proof

- $$\frac{\partial}{\partial t} f_{N,M}^{(N,0)}(V,t) = \lambda_1 \sum_{j=1}^{N} \Gamma_{N,0} \int_{\Omega_{N,0}} \left(\overline{F}(V_{w_1}^j, t) - \overline{F}(V,t) \right) \, d\sigma(W)$$
$$+ \lambda_1 \sum_{j=1}^{N} \Gamma_{N,0} \int_{\Omega_{N,0}} \left(F(V_{w_1}^j, W_{v_j}^1, t) - \overline{F}(V_{w_1}^j, t) \right) \, d\sigma(W)$$

- An estimate for the second term:

$$\left| \int_{B_N} I_2 g(v_1) \, dV \right|^2 \le \left(2\lambda_1^2 \|g\|_\infty^2 e^{-tN\lambda_2} + \frac{\lambda_1^2 \|g\|_\infty^2}{\lambda_2^2} \right) \int_{\Omega_{0,0}} |F(V,W,0)|^2 \, d\sigma(V,W)$$

Bonetto et al, arXiv 2016

- 

- N particles $V = (v_1, ..., v_N)$
- M "thermostat" particles $W = (w_1, ..., w_M)$
- Exchange between active and passive as before
- Pairwise collisions between V-particles *and* W-particles
- $\|e^{\bar{\mathcal{L}}t}h_0 - e^{\mathcal{L}t}h_0\|_2 \leq \frac{N}{\sqrt{M}}(1 - e^{-\mu t/2})\|h_0 - 1\|_2$

- Similar resultat in Fourier metric

A Kac model for relativistic particles

- Classical: Jump process on $S^{N-1}(\sqrt{N})$.
- Average energy: $\dfrac{1}{N}\sum_{j=1}^{N} v_j^2$.
- Invariant density: Uniform on $S^{N-1}(\sqrt{N})$

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- Much less symmetric
- Invariant measures less obvious.

The micro-canonical measure

- Let $H(v_1, \ldots, v_N) = \sum_{j=1}^{N} \phi(v_j)$
- The micro-canonical measure on $\Omega^N(\sqrt{N})$ is

$$\eta^{(N)} = \frac{\sigma_\Omega}{|\nabla H|}$$

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- As distributions:

$$\int_{\mathbb{R}^N} g(v_1, \ldots, v_N)\delta(H(v_1, \ldots, v_N) - E)dv_1 \ldots dv_N$$
$$= \int_{\mathbb{R}} \int_{\Omega^{N-1}(\sqrt{y})} g\delta(y - E)\frac{d\sigma_\Omega}{|\nabla H|}dy = \int_{\Omega^{N-1}(\sqrt{E})} g\frac{d\sigma_\Omega}{|\nabla H|}.$$

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- This shows that

$$\delta(H(v_1,\dots,v_N)-E) = \frac{\sigma_\Omega}{|\nabla H|}$$

Marginals of distributions on $\Omega^N(\sqrt{N})$: a Theorem

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- Then the uniform densities w.r.t. the micro-canonical measures on are

$$Ce^{-z_0\phi(v)} \qquad \text{- chaotic}$$

- The positive constant z_0 is the unique real solution to the following equation

$$\int_{-\infty}^{\infty}(1-\phi(v))e^{-z_0\phi(v)}dv = 0,$$

The assumptions

- $\phi \in C^1(\mathbb{R}, \mathbb{R}^+)$, ϕ is even and convex

  $\phi(0) = 0$

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- $\phi \in C^1(\mathbb{R}, \mathbb{R}^+)$, ϕ is even and convex
  $\phi(0) = 0$
- Let $f(y) := \dfrac{|y|}{|\phi'(\phi^{-1}(y^2))|}$. We need:
- $\displaystyle\int_{-\infty}^{\infty} e^{-\gamma y^2} f(y)\,dy < \infty.$
- $\displaystyle\int_{-\infty}^{\infty} (1 - y^2) f(y)\,dy < 0$
- $\displaystyle\int_{|y|\le 1} (1 - y^2) f(y)\,dy > 0$

- Need to compute

$$Z_\phi(\sqrt{N}) = \int_{\Omega^{N-1}(\sqrt{E})} d\eta^{(N)},$$

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$$2^N \int_{\sum_{i=1}^{N-1} y_i^2 \leq E} \frac{1}{\left|\phi'(\phi^{-1}(E - \sum_{i=1}^{N-1} y_i^2))\right|} \prod_{i=1}^{N-1} \frac{|y_i|}{|\phi'(\phi^{-1}(y_i^2))|} dy_1 \dots dy_{N-1}.$$

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- Proof completed by the saddle point method
- Conditions needed to guarantee a unique saddle point

Result in higher dimension

- For $p \in \mathbb{R}^2$, define

$$\Gamma^N(\sqrt{E}, p) = \left\{ (v_1, \ldots, v_N) \in \mathbb{R}^{2N} \;\middle|\; \sum_{i=1}^{N} \phi(v_i) = 2E, \; \sum_{i=1}^{N} v_i = p \right\}.$$

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$$\mu_{E,p_1,p_2} = \delta(2E - \sum_{i=1}^{N} \phi(v_i))\delta(p_1 - \sum_{i=1}^{N} v_{i1})\delta(p_2 - \sum_{i=1}^{N} v_{i2}).$$

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- The uniform densities w.r.t. μ_{N,p_1,p_2} are

$Ce^{-z_0\phi(v)}$ -chaotic

- (but only formal calculations)

Thank you for listening

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and thanks to

- Pierre Degond
- Eric Carlen
- Maria Carvalo