On the quasineutral limit of the Vlasov-Poisson system

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Vlasov-Poisson with small Debye length

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \quad t \geq 0, \, (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \\ E_\varepsilon = -\nabla_x U_\varepsilon, \\ U_\varepsilon - \varepsilon^2 \Delta_x U_\varepsilon = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon \, dv - 1, \\ f_\varepsilon|_{t=0} = f_{0,\varepsilon} \geq 0, \; \displaystyle\int_{\mathbb{T}^d \times \mathbb{R}^d} f_{0,\varepsilon} \, dv \, dx = 1. \end{cases}$$

• f_ε describes the dynamics of ions, in a background of massless electrons following a linearized Maxwell-Boltzmann law :

$$n_{\mathrm{elec}} = e^{U_\varepsilon} \sim 1 + U_\varepsilon.$$

• The parameter $\varepsilon \in (0, 1]$ is the ratio between the **Debye length** and the observation length. In practice, $\varepsilon \ll 1$.

• **Quasineutral limit** : $\varepsilon \to 0$.

Vlasov-Dirac-Benney

Assuming $f_{0,\varepsilon} \to f_0$ and taking $\varepsilon = 0$ yields

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ U = \displaystyle\int_{\mathbb{R}^d} f \, dv - 1, \\ f|_{t=0} = f_0 \geq 0, \quad \displaystyle\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dvdx = 1. \end{cases}$$

- ... a system called **Vlasov-Dirac-Benney** by Bardos.
- **Loss of derivative ?** The force E is one derivative less regular than f.
- Is Vlasov-Dirac-Benney a good approximation of Vlasov-Poisson when $\varepsilon \to 0$?

More on Vlasov-Dirac-Benney : ill-posedness

• There are **equilibria** around which the **linearized** equations have **unbounded unstable spectrum** [Bardos, Nouri 2012].
Examples of such **unstable** equilibria are homogeneous **two bumps profiles** (hence the usual name **two stream instabilities**).

• Consequence : **Vlasov-Dirac-Benney is not well-posed in the sense of Hadamard**.
The solution map is not Hölder continuous, even with arbitrary loss of derivatives and for arbitrarily small times [DHK, T. Nguyen 2016].

(Same abstract argument holds for the **hydrostatic Euler** equations.)

More on Vlasov-Dirac-Benney : well-posedness

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f = 0, \\ \rho = \displaystyle\int_{\mathbb{R}^d} f \, dv, \\ f|_{t=0} = f_0 \geq 0, \quad \displaystyle\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dv dx = 1. \end{cases}$$

(Local) Existence of solutions is known for

- **analytic** initial data (Cauchy-Kowalevski type result) ;
- in $d = 1$, Sobolev initial data that, for all x, are **one bump profiles** [Bardos, Besse 2013], through a hydrodynamic representation ;
- **Penrose stable Sobolev** initial data [DHK, Rousset, to appear]. More to come in a few minutes !

• The change of variables $(t, x, v) \mapsto (\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v)$ gives the **unscaled** Vlasov-Poisson system (that is, with $\varepsilon = 1$).

QUASINEUTRAL LIMIT $\rightarrow$ **LARGE TIME BEHAVIOR** problem

• What matters :
STABILITY or INSTABILITY of (homogeneous) equilibria

Consequences of instability in the quasineutral limit

• VDB is **not a good approximation** for initial data polynomially close to **unstable** equilibria $\mu(v)$ such as two bumps profiles [DHK, Hauray 2015]. One finds for all $n, m, k \geq 0$ a family of solutions (f_ε) such that

$$\|f_\varepsilon(0) - \mu\|_{W^{n,1}_{x,v}} \leq \varepsilon^k, \qquad \text{but}$$

$$\|f_\varepsilon(T_\varepsilon) - \mu\|_{W^{-m,1}_{x,v}} > 0, \quad \text{for } T_\varepsilon = O(\varepsilon|\log \varepsilon|).$$

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• In [Grenier '96], it is shown that these instabilities have no effect for solutions with **uniform analytic regularity** $\implies$ **Derivation of VDB**.

[DHK, Iacobelli, preprints] : still true for **exponentially small but rough** perturbations of such data ($d \leq 3$). See Mikaela's talk tomorrow !

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• What about **stable** cases ?

Stable case : monokinetic data

• Note that $f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}$ satisfies VDB iff (ρ, u) satisfies the **Shallow Water equations**.

Dirac measure : zero-temperature limit, extremal case of a Maxwellian.

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• Modulated energy method [Brenier 2000] : introduce as in [DHK 2011]

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v - u(t,x)|^2\, dvdx + \int \frac{(U_\varepsilon - \rho(t,x))^2}{2}\, dx + \frac{\varepsilon^2}{2}\int |E_\varepsilon(t,x)|^2\, dx.$$

and show $\mathcal{H}_\varepsilon$ is a Lyapunov functional, controlling the distance to the Dirac measure $\implies$ **Derivation of Shallow Water**.

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• This **method requires that the target is the minimizer of some entropy**, and thus satisfies **symmetry** properties in v.
From [DHK, Hauray 2015], one can not find regular solutions of VDB satisfying these constraints, except **symmetric homogeneous equilibria**.

We thus cannot handle general **stable** initial data with the modulated energy method.

Derivation result for stable data

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- We say that $f_0(v)$ satisfies the c_0 **Penrose stability condition** if

$$\inf_{(\gamma,\tau,\eta)\in\mathbb{R}_*^+\times\mathbb{R}\times\mathbb{R}_*^d}\left|1-\int_0^{+\infty} e^{-(\gamma+i\tau)s}\,\frac{i\eta}{1+|\eta|^2}\cdot\int_{\mathbb{R}^d} e^{-is\eta\cdot v}(\nabla_v f_0)(v)\,dv\,ds\right| \geq c_0.$$

Same as for **Landau Damping** [Mouhot, Villani 2011].
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- Introduce also for $k\in\mathbb{N}$, $r\in\mathbb{R}$, the weighted Sobolev norms

$$\|f\|_{\mathcal{H}_r^k} := \left(\sum_{|\alpha|+|\beta|\leq k}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(1+|v|^2)^r|\partial_x^\alpha\partial_v^\beta f|^2\,dvdx\right)^{1/2}.$$

Derivation result for stable data

Theorem 1 (*[DHK, Rousset, to appear]*)

Let m, r be large enough. Let $M_0 > 0, c_0 > 0$. Assume that for all $\varepsilon \in (0,1]$, $\|f_{0,\varepsilon}\|_{\mathcal{H}^{2m}_{2r}} \leq M_0$ and for all $x \in \mathbb{T}^d$, $f_{0,\varepsilon}(x,\cdot)$ satisfies the c_0 Penrose stability condition. Assume that $f_{0,\varepsilon} \to f_0$ in L^2. Then there is $T > 0$ such that

$$\sup_{[0,T]} \|f_\varepsilon(t) - f(t)\|_{L^2} \to 0,$$

where $f(t)$ satisfies Vlasov-Dirac-Benney with initial data f_0.

Derivation result for stable data

Theorem 2 (*[DHK, Rousset, to appear]*)

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An example of admissible initial data is given by smooth local Maxwellians

$$M(x,v) = \frac{\rho(x)}{(2\pi T(x))^{d/2}} \exp\left(-\frac{|v-u(x)|^2}{T(x)}\right).$$

As a by-product we get **well-posedness (i.e. existence + uniqueness) in the class of such data for Vlasov-Dirac-Benney**.

Derivation result for stable data

Theorem 3 (*[DHK, Rousset, to appear]*)

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Sketch of the proof

We have $\|f_{0,\varepsilon}\|_{\mathcal{H}^{2m}_{2r}} \leq M_0$. Introduce

$$\boxed{\mathcal{N}_{2m,\,2r}(t, f_\varepsilon) := \|f_\varepsilon\|_{L^\infty((0,t),\mathcal{H}^{2m-1}_{2r})} + \|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}}$$

with $\rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, dv$. The main task is to find $T > 0$, $R > 0$ such that

$$\forall \varepsilon \in (0,1], \quad \sup_{[0,T]} \mathcal{N}_{2m,\,2r}(t, f_\varepsilon) \leq R.$$

The proof is based on a bootstrap argument.
By a standard energy estimate, we see that the key quantity to be controlled is actually $\|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}$.

Sketch of the proof

• Let us look for an estimate for $\|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}$.

• **Natural idea** : up to commutators, $\partial_{\mathbf{x}}^{\mathbf{2m}}\mathbf{f}_\varepsilon$ evolves according to the linearized equation around f_ε, that is

$$\partial_t \partial_{\mathbf{x}}^{\mathbf{2m}}\mathbf{f}_\varepsilon + v \cdot \nabla_x \partial_{\mathbf{x}}^{\mathbf{2m}}\mathbf{f}_\varepsilon + \partial_{\mathbf{x}}^{\mathbf{2m}}\mathbf{E}_\varepsilon \cdot \nabla_v f_\varepsilon + E_\varepsilon \cdot \nabla_v \partial_{\mathbf{x}}^{\mathbf{2m}}\mathbf{f}_\varepsilon = S,$$

where S should involve remainder terms only.

• When $f_\varepsilon \equiv \mu(v)$ does not depend on t and x, then $E_\varepsilon = 0$ and the linearized equation reduces to

$$\begin{aligned} &(\partial_t + v \cdot \nabla_x)\, g + E_g \cdot \nabla_v \mu(v) = S, \\ &E_g = -\nabla_x (I - \varepsilon^2 \Delta)^{-1} \rho_g. \end{aligned}$$

This case was studied in [Mouhot, Villani 2011] in view of Landau Damping.

Sketch of the proof

By the method of characteristics,

$$g(t,x,v) = g_0(x - tv, v) - \int_0^t E_g(s, x-(t-s)v) \cdot \nabla_v \mu(v)\, ds + \mathcal{S}$$

and thus, integrating in v, we obtain an integral equation for $\rho_g = \int_{\mathbb{R}^d} g\, dv$:

$$\rho_g(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x (I - \varepsilon^2 \Delta)^{-1} \rho_g(s, x-(t-s)v) \cdot \nabla_v \mu(v)\, dvds + \mathcal{S} + \mathcal{S}_0$$

By **Fourier analysis** (in x and t), one may estimate ρ_g in $L^2_{t,x}$, **under the Penrose stability condition** for $\mu(v)$.

Sketch of the proof

Let us thus study (morally $g = \partial_x^{2m} f_\varepsilon$)

$$(\partial_t + v \cdot \nabla_x + E_\varepsilon \cdot \nabla_v) \, g + E_g \cdot \nabla_v f_\varepsilon = S,$$

$$E_g := -\nabla_x (I - \varepsilon^2 \Delta_x)^{-1} \int_{\mathbb{R}^d} g \, dv.$$

As f_ε depends on x, the force field E_ε is not trivial.

However, by the (near identity for small times) change of variables

$$v \mapsto \Phi(t, x, v), \qquad \partial_t \Phi + \Phi \cdot \nabla_x \Phi = E_\varepsilon, \quad \Phi|_{t=0} = v,$$

it comes down to

$$(\partial_t + \Phi(t, x, v) \cdot \nabla_x) \, g + E_g \cdot \nabla_v f_\varepsilon = S.$$

Integrating along characteristics and integrating in v, we end up with the study, for **small times**, of...

...the integral equation on $h = \int_{\mathbb{R}^d} g \, dv$:

$$h = K_{\nabla_v f_{0,\varepsilon}} (I - \varepsilon^2 \Delta)^{-1} h + R,$$

with

$$K_{\nabla_v f_{0,\varepsilon}}(h) = \int_0^t \int_{\mathbb{R}^d} (\nabla_{\mathbf{x}} h)(s, x - (t-s)v) \cdot \nabla_v f_{0,\varepsilon}(x,v) \, dv \, ds.$$

Note that $K_{\nabla_v f_{0,\varepsilon}}$ may seem to feature a loss of derivative. However, we have

Proposition 1

$K_{\nabla_v f_0}$ is a bounded operator on L^2 if f_0 is smooth and decaying in v.

This is an effect in the spirit of **averaging lemmas** ([Golse, Lions, Perthame, Sentis '88]).

Sketch of the proof

$$h = K_{\nabla_v f_{0,\varepsilon}} (I - \varepsilon^2 \Delta)^{-1} h + R$$

Let $\gamma > 0$ to be chosen. We can relate $e^{-\gamma t} K_{\nabla_v f_{0,\varepsilon}} (e^{\gamma t} (I - \varepsilon^2 \Delta)^{-1} \cdot)$ to a **semi-classical pseudodifferential operator**, of symbol

$$a(\gamma, \tau, x, \eta) := \int_0^{+\infty} e^{-(\gamma + i\tau)s} \frac{i\eta}{1 + |\eta|^2} \cdot \int_{\mathbb{R}^d} e^{-is\eta \cdot v} \nabla_v f_{0,\varepsilon}(x, v) \, dv \, ds,$$

and we rewrite the integral equation as

$$Op^{\gamma,\varepsilon}_{(1-a)}(e^{-\gamma t} h) = \mathcal{R}.$$

The c_0 Penrose condition is precisely $\inf |1 - a| \geq c_0$ and therefore implies the **ellipticity of the symbol** associated to the operator we want to invert.

Sketch of the proof

$$Op^{\gamma,\varepsilon}_{(1-a)}(e^{-\gamma t}h) = \mathcal{R}.$$

We can finally use a **semi-classical pseudodifferential calculus with parameter** γ in order to invert $Op^{\gamma,\varepsilon}_{(1-a)}$ up to a **small remainder**.
More precisely we have the general estimate

$$\| Op^{\gamma,\varepsilon}_b Op^{\gamma,\varepsilon}_c u - Op^{\gamma,\varepsilon}_{bc} u\|_{L^2} \lesssim \frac{1}{\gamma}\|b\|\|c\|\|u\|_{L^2},$$

that we apply to

$$b = \frac{1}{1-a}, \quad c = 1-a, \quad u = e^{-\gamma t}h,$$

which roughly gives

$$\|h\|_{L^2} \leq \|Op^{\gamma,\varepsilon}_{\frac{1}{1-a}}\mathcal{R}\|_{L^2} + \frac{1}{\gamma}\left\|\frac{1}{1-a}\right\| \|1-a\|\|h\|_{L^2}.$$

Choosing $\gamma \gg 1$ yields an estimate for h, in $L^2_{t,x}$.
This allows to close the bootstrap argument. That's it !