On the quasineutral limit of the Vlasov-Poisson system

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Vlasov-Poisson with small Debye length

\[
\begin{aligned}
\begin{cases}
\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, & t \geq 0, (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \\
E_\varepsilon = -\nabla_x U_\varepsilon, \\
U_\varepsilon - \varepsilon^2 \Delta_x U_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, dv - 1, \\
f_\varepsilon |_{t=0} = f_{0,\varepsilon} \geq 0, & \int_{\mathbb{T}^d \times \mathbb{R}^d} f_{0,\varepsilon} \, dv \, dx = 1.
\end{cases}
\end{aligned}
\]

• \(f_\varepsilon\) describes the dynamics of ions, in a background of massless electrons following a linearized Maxwell-Boltzmann law:

\[n_{\text{elec}} = e^{U_\varepsilon} \sim 1 + U_\varepsilon.\]

• The parameter \(\varepsilon \in (0, 1]\) is the ratio between the Debye length and the observation length. In practice, \(\varepsilon \ll 1\).

• Quasineutral limit : \(\varepsilon \to 0\).
Vlasov-Dirac-Benney

Assuming $f_{0,\varepsilon} \to f_0$ and taking $\varepsilon = 0$ yields

$$
\begin{cases}
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\
E = -\nabla_x U, \\
U = \int_{\mathbb{R}^d} f \, dv - 1, \\
f|_{t=0} = f_0 \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dvdx = 1.
\end{cases}
$$

- ... a system called Vlasov-Dirac-Benney by Bardos.
- **Loss of derivative?** The force $E$ is one derivative less regular than $f$.
- Is Vlasov-Dirac-Benney a good approximation of Vlasov-Poisson when $\varepsilon \to 0$?
• There are equilibria around which the linearized equations have unbounded unstable spectrum [Bardos, Nouri 2012]. Examples of such unstable equilibria are homogeneous two bumps profiles (hence the usual name two stream instabilities).

• Consequence: Vlasov-Dirac-Benney is not well-posed in the sense of Hadamard. The solution map is not Hölder continuous, even with arbitrary loss of derivatives and for arbitrarily small times [DHK, T. Nguyen 2016].

(Same abstract argument holds for the hydrostatic Euler equations.)
More on Vlasov-Dirac-Benney: well-posedness

\[
\begin{aligned}
\partial_t f + \mathbf{v} \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f &= 0, \\
\rho &= \int_{\mathbb{R}^d} f \, dv,
\end{aligned}
\]

\[
\begin{aligned}
f|_{t=0} &= f_0 \geq 0, \\
\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dv \, dx &= 1.
\end{aligned}
\]

(Local) Existence of solutions is known for

- **analytic** initial data (Cauchy-Kowalevski type result);

- in \( d = 1 \), Sobolev initial data that, for all \( x \), are **one bump profiles** [Bardos, Besse 2013], through a hydrodynamic representation;

- **Penrose stable Sobolev** initial data [DHK, Rousset, to appear].

More to come in a few minutes!
The change of variables \((t, x, v) \mapsto (\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v)\) gives the unscaled Vlasov-Poisson system (that is, with \(\varepsilon = 1\)).

QUASINEUTRAL LIMIT \(\rightarrow\) LARGE TIME BEHAVIOR problem

What matters:
STABILITY or INSTABILITY of (homogeneous) equilibria
• VDB is not a good approximation for initial data polynomially close to unstable equilibria $\mu(v)$ such as two bumps profiles [DHK, Hauray 2015]. One finds for all $n, m, k \geq 0$ a family of solutions $(f_\varepsilon)$ such that

$$\|f_\varepsilon(0) - \mu\|_{W_{x,v}^{n,1}} \leq \varepsilon^k,$$

but

$$\|f_\varepsilon(T_\varepsilon) - \mu\|_{W_{x,v}^{-m,1}} > 0,$$

for $T_\varepsilon = O(\varepsilon|\log\varepsilon|)$. 

• In [Grenier '96], it is shown that these instabilities have no effect for solutions with uniform analytic regularity $\Rightarrow$ Derivation of VDB. [DHK, Iacobelli, preprints]: still true for exponentially small but rough perturbations of such data ($d \leq 3$). See Mikaela's talk tomorrow!
Consequences of instability in the quasineutral limit

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Consequences of instability in the quasineutral limit

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• What about **stable** cases?
Stable case: monokinetic data

• Note that \( f(t, x, v) = \rho(t, x)\delta_{v=u(t,x)} \) satisfies VDB iff \((\rho, u)\) satisfies the\nShallow Water equations.\n
Dirac measure: zero-temperature limit, extremal case of a Maxwellian.
Stable case: monokinetic data

- Note that \( f(t, x, v) = \rho(t, x)\delta_{v= u(t,x)} \) satisfies VDB iff \((\rho, u)\) satisfies the Shallow Water equations.

Dirac measure: zero-temperature limit, extremal case of a Maxwellian.

- **Modulated energy method [Brenier 2000]:** introduce as in [DHK 2011]

\[
\mathcal{H}_\varepsilon(t) = \frac{1}{2} \int f\varepsilon |v - u(t, x)|^2 \, dv \, dx + \int \frac{(U\varepsilon - \rho(t, x))^2}{2} \, dx + \varepsilon^2 \int |E\varepsilon(t, x)|^2 \, dx.
\]

and show \( \mathcal{H}_\varepsilon \) is a Lyapunov functional, controlling the distance to the Dirac measure \( \implies \) **Derivation of Shallow Water.**
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\]

and show \( \mathcal{H}_\varepsilon \) is a Lyapunov functional, controlling the distance to the Dirac measure \( \Rightarrow \) Derivation of Shallow Water.

- This method requires that the target is the minimizer of some entropy, and thus satisfies symmetry properties in \( v \).

From [DHK, Hauray 2015], one can not find regular solutions of VDB satisfying these constraints, except symmetric homogeneous equilibria.

We thus cannot handle general stable initial data with the modulated energy method.
Another method is needed to handle general stable data. That’s the purpose of a recent joint work with F. Rousset.
Derivation result for stable data

Another method is needed to handle general stable data. That’s the purpose of a recent joint work with F. Rousset.

- We say that $f_0(v)$ satisfies the $c_0$ Penrose stability condition if

$$\inf_{(\gamma, \tau, \eta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d} \left| 1 - \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{i\eta}{1+|\eta|^2} \cdot \int_{\mathbb{R}^d} e^{-is\eta \cdot v} (\nabla_v f_0)(v) \, dv \, ds \right| \geq c_0.$$ 

Same as for Landau Damping [Mouhot, Villani 2011]. (includes in particular one bump profiles)
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- Introduce also for \( k \in \mathbb{N}, r \in \mathbb{R} \), the weighted Sobolev norms

\[
\| f \|_{\mathcal{H}_r^k} := \left( \sum_{|\alpha| + |\beta| \leq k} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (1 + |v|^2)^r |\partial_x^\alpha \partial_v^\beta f|^2 \, dv \, dx \right)^{1/2}.
\]
Theorem 1 ([DHK, Rousset, to appear])

Let $m, r$ be large enough. Let $M_0 > 0, c_0 > 0$. Assume that for all $\varepsilon \in (0, 1]$, $\|f_{0,\varepsilon}\|_{H^{2m}} \leq M_0$ and for all $x \in \mathbb{T}^d$, $f_{0,\varepsilon}(x, \cdot)$ satisfies the $c_0$ Penrose stability condition. Assume that $f_{0,\varepsilon} \to f_0$ in $L^2$. Then there is $T > 0$ such that

$$\sup_{[0, T]} \|f_{\varepsilon}(t) - f(t)\|_{L^2} \to 0,$$

where $f(t)$ satisfies Vlasov-Dirac-Benney with initial data $f_0$. 

An example of admissible initial data is given by smooth local Maxwellians $M(x, v) = \rho(x) (2\pi T(x))^d/2 \exp(-|v - u(x)|^2 T(x))$. 

As a by-product we get well-posedness (i.e. existence + uniqueness) in the class of such data for Vlasov-Dirac-Benney.
Theorem 2 ([DHK, Rousset, to appear])

Let $m, r$ be large enough. Let $M_0 > 0$, $c_0 > 0$. Assume that for all $\varepsilon \in (0, 1]$, $\|f_{0,\varepsilon}\|_{\mathcal{H}^{2m}} \leq M_0$ and for all $x \in \mathbb{T}^d$, $f_{0,\varepsilon}(x, \cdot)$ satisfies the $c_0$ Penrose stability condition. Assume that $f_{0,\varepsilon} \to f_0$ in $L^2$. Then there is $T > 0$ such that

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$$M(x, v) = \frac{\rho(x)}{(2\pi T(x))^{d/2}} \exp \left( -\frac{|v - u(x)|^2}{T(x)} \right).$$

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Theorem 3 ([DHK, Rousset, to appear])

Let $m, r$ be large enough. Let $M_0 > 0$, $c_0 > 0$. Assume that for all $\varepsilon \in (0, 1]$, $\|f_{0,\varepsilon}\|_{\mathcal{H}^{2m}} \leq M_0$ and for all $\mathbf{x} \in \mathbb{T}^d$, $f_{0,\varepsilon}(\mathbf{x}, \cdot)$ satisfies the $c_0$ Penrose stability condition. Assume that $f_{0,\varepsilon} \to f_0$ in $L^2$. Then there is $T > 0$ such that

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As a by-product we get well-posedness (i.e. existence + uniqueness) in the class of such data for Vlasov-Dirac-Benney.
Sketch of the proof

We have \( \| f_{0, \varepsilon} \|_{H^{2m}_{2r}} \leq M_0 \). Introduce

\[
N_{2m, 2r}(t, f_{\varepsilon}) := \| f_{\varepsilon} \|_{L^\infty((0, t), H^{2m-1}_{2r})} + \| \rho_{\varepsilon} \|_{L^2((0, t), H^{2m})}
\]

with \( \rho_{\varepsilon} = \int_{\mathbb{R}^d} f_{\varepsilon} \, dv \). The main task is to find \( T > 0, R > 0 \) such that

\[
\forall \varepsilon \in (0, 1], \quad \sup_{[0, T]} N_{2m, 2r}(t, f_{\varepsilon}) \leq R.
\]

The proof is based on a bootstrap argument. By a standard energy estimate, we see that the key quantity to be controlled is actually \( \| \rho_{\varepsilon} \|_{L^2((0, t), H^{2m})} \).
Sketch of the proof

• Let us look for an estimate for $\|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}$.

• **Natural idea**: up to commutators, $\partial_x^{2m}f_\varepsilon$ evolves according to the linearized equation around $f_\varepsilon$, that is

$$
\partial_t \partial_x^{2m}f_\varepsilon + \nu \cdot \nabla_x \partial_x^{2m}f_\varepsilon + \partial_x^{2m}E_\varepsilon \cdot \nabla_\nu f_\varepsilon + E_\varepsilon \cdot \nabla_\nu \partial_x^{2m}f_\varepsilon = S,
$$

where $S$ should involve remainder terms only.

• When $f_\varepsilon \equiv \mu(\nu)$ does not depend on $t$ and $x$, then $E_\varepsilon = 0$ and the linearized equation reduces to

$$
(\partial_t + \nu \cdot \nabla_x) g + E_g \cdot \nabla_\nu \mu(\nu) = S,
$$

$$
E_g = -\nabla_x (I - \varepsilon^2 \Delta)^{-1} \rho_g.
$$

This case was studied in [Mouhot, Villani 2011] in view of Landau Damping.
Sketch of the proof

By the method of characteristics,

\[ g(t, x, v) = g_0(x - tv, v) - \int_0^t E_g(s, x - (t - s)v) \cdot \nabla_v \mu(v) \, ds + S \]

and thus, integrating in \( v \), we obtain an integral equation for \( \rho_g = \int_{\mathbb{R}^d} g \, dv \):

\[ \rho_g(t, x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x (I - \varepsilon^2 \Delta)^{-1} \rho_g(s, x - (t - s)v) \cdot \nabla_v \mu(v) \, dv \, ds + S + S_0 \]

By Fourier analysis (in \( x \) and \( t \)), one may estimate \( \rho_g \) in \( L^2_{t, x} \), under the Penrose stability condition for \( \mu(v) \).
Sketch of the proof

Let us thus study (morally $g = \partial_x^2 \epsilon f$)

$$(\partial_t + v \cdot \nabla_x + E_\epsilon \cdot \nabla_v) g + E_g \cdot \nabla_v f_\epsilon = S,$$

$$E_g := -\nabla_x (I - \epsilon^2 \Delta_x)^{-1} \int_{\mathbb{R}^d} g \, dv.$$ 

As $f_\epsilon$ depends on $x$, the force field $E_\epsilon$ is not trivial.

However, by the (near identity for small times) change of variables

$$v \mapsto \Phi(t, x, v), \quad \partial_t \Phi + \Phi \cdot \nabla_x \Phi = E_\epsilon, \quad \Phi|_{t=0} = v,$$

it comes down to

$$(\partial_t + \Phi(t, x, v) \cdot \nabla_x) g + E_g \cdot \nabla_v f_\epsilon = S.$$

Integrating along characteristics and integrating in $v$, we end up with the study, for small times, of...
...the integral equation on $h = \int_{\mathbb{R}^d} g \, dv$:

$$h = K_{\nabla v f_0, \varepsilon} (I - \varepsilon^2 \Delta)^{-1} h + R,$$

with

$$K_{\nabla v f_0, \varepsilon} (h) = \int_0^t \int_{\mathbb{R}^d} (\nabla_x h)(s, x - (t - s)v) \cdot \nabla v f_0, \varepsilon(x, v) \, dv \, ds.$$

Note that $K_{\nabla v f_0, \varepsilon}$ may seem to feature a loss of derivative. However, we have

**Proposition 1**

$K_{\nabla v f_0}$ is a bounded operator on $L^2$ if $f_0$ is smooth and decaying in $v$.

This is an effect in the spirit of **averaging lemmas** ([Golse, Lions, Perthame, Sentis '88]).
Sketch of the proof

\[ h = K_{\nabla v f_0, \varepsilon} (I - \varepsilon^2 \Delta)^{-1} h + R \]

Let \( \gamma > 0 \) to be chosen. We can relate \( e^{-\gamma t} K_{\nabla v f_0, \varepsilon} (e^{\gamma t} (I - \varepsilon^2 \Delta)^{-1}) \) to a semi-classical pseudodifferential operator, of symbol

\[
a(\gamma, \tau, x, \eta) := \int_0^{+\infty} e^{-(\gamma + i\tau)s} \frac{i\eta}{1 + |\eta|^2} \cdot \int_{\mathbb{R}^d} e^{-i\eta \cdot v} \nabla v f_{0, \varepsilon}(x, v) \, dv \, ds,
\]

and we rewrite the integral equation as

\[
Op^{\gamma, \varepsilon}_{(1-a)}(e^{-\gamma t} h) = \mathcal{R}.
\]

The \( c_0 \) Penrose condition is precisely \( \inf |1 - a| \geq c_0 \) and therefore implies the ellipticity of the symbol associated to the operator we want to invert.
Sketch of the proof

\[ \text{Op}_{(1-a)}^{\gamma, \varepsilon}(e^{-\gamma t} h) = \mathcal{R}. \]

We can finally use a **semi-classical pseudodifferential calculus with parameter** \( \gamma \) in order to invert \( \text{Op}_{(1-a)}^{\gamma, \varepsilon} \) up to a **small remainder**. More precisely we have the general estimate

\[
\| \text{Op}_{b}^{\gamma, \varepsilon} \text{Op}_{c}^{\gamma, \varepsilon} u - \text{Op}_{bc}^{\gamma, \varepsilon} u \|_{L^2} \lesssim \frac{1}{\gamma} \| b \| \| c \| \| u \|_{L^2},
\]

that we apply to

\[
b = \frac{1}{1-a}, \quad c = 1-a, \quad u = e^{-\gamma t} h,
\]

which roughly gives

\[
\| h \|_{L^2} \leq \| \text{Op}_{1-a}^{\gamma, \varepsilon} \mathcal{R} \|_{L^2} + \frac{1}{\gamma} \left\| \frac{1}{1-a} \right\| \| 1-a \| \| h \|_{L^2}.
\]

Choosing \( \gamma \gg 1 \) yields an estimate for \( h \), in \( L^2_{t,x} \).

This allows to close the bootstrap argument. That’s it!