Quasineutral limit for Vlasov-Poisson via Wasserstein stability estimates

Mikaela Iacobelli

University of Cambridge

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The Vlasov-Poisson system

The Vlasov-Poisson system is a classical model used to describe the evolution of plasmas.

In its most common form, $f(t,x,v)$ represents the distribution function for the electrons, while ions are assumed to be at rest, and to act as a fixed background.

The VP equation on the torus $\mathbb{T}^d$ is given by

$$(VPE)\begin{cases} \partial_t f + v\cdot\nabla_x f + E(t,x)\cdot\nabla_v f = 0, \\ E = -\nabla_x U, \quad \rho(t,x) = \int f(t,x,v)\,dv \\ -\Delta U = \rho - 1 \\ f|_{t=0} = f_0 \geq 0, \quad \int_{\mathbb{T}^d\times\mathbb{R}^d} f_0\,dx\,dv = 1. \end{cases}$$

The massless electrons case

There are several variants of this equation.

For instance, from the point of view of ions, one can assume that electrons have zero mass and they reach their local thermodynamic equilibrium quasi-instantaneously.

Then, the density of electrons follows the Maxwell-Boltzmann law, and the Poisson equation takes the form

$$-\Delta U = \rho - e^U.$$

Debye length

The Debye length λ_D can be interpreted as the typical length below which charge separation occurs.

In plasmas, typical values go from 10^{-3} m to 10^{-8} m. Hence, in dimensionless variables, the Poisson equation reads

$$-\varepsilon^2 \Delta U = \rho - e^U,$$

where $\varepsilon \approx \lambda_D$.

The **quasineutral limit** consists in considering the limit $\varepsilon \to 0$.

Some results on quasineutral limit

- Brenier 1989, 2000 $\rightsquigarrow$ well prepared monokinetic data (i.e. cold electrons)
- Grenier 1995,1996,1999 $\rightsquigarrow$ convergence for analytic initial data
- Masmoudi 2001 $\rightsquigarrow$ general monokinetic data
- Golse - Saint-Raimond 2003 $\rightsquigarrow$ well prepared data with magnetic field
- Han-Kwan - Hauray 2014 $\rightsquigarrow$ stability-instability issue
- Han-Kwan - Rousset 2016 $\rightsquigarrow$ convergence for some Sobolev perturbations

The massless electron model

The Vlasov-Poisson system with massless electrons reads as follows:

$$(VPME) := \begin{cases} \partial_t f + v \cdot \partial_x f + E \cdot \partial_v f = 0, \\ E = -U', \\ U'' = e^U - \rho, \\ f|_{t=0} = f_0 \geq 0, \quad \int f_0 \, dx \, dv = 1. \end{cases} \tag{1}$$

Because of the additional semilinear term in the Poisson equation, this system is more complicated with respect to the classical Vlasov-Poisson system.

The quasineutral limit leads to the study of the limit as $\varepsilon \to 0$ of the scaled system

$$(VPME)_\varepsilon := \begin{cases} \partial_t f_\varepsilon + v \cdot \partial_x f_\varepsilon + E_\varepsilon \cdot \partial_v f_\varepsilon = 0, \\ E_\varepsilon = -U_\varepsilon', \\ \varepsilon^2 U_\varepsilon'' = e^{U_\varepsilon} - \rho_\varepsilon, \\ f_\varepsilon|_{t=0} = f_{0,\varepsilon} \geq 0, \quad \int f_{0,\varepsilon}\, dx\, dv = 1. \end{cases} \tag{2}$$

Formal limit taking $\varepsilon = 0$: the *kinetic isothermal Euler system*.

$$(KIE) := \begin{cases} \partial_t f + v \cdot \partial_x f + E \cdot \partial_v f = 0, \\ E = -U', \\ U = \log \rho, \\ f_0 \geq 0, \quad \int f_0\, dx\, dv = 1. \end{cases} \tag{3}$$

The justification of this convergence is highly nontrivial: this was proved by Grenier under the assumptions that the initial data are bounded in the analytic norm

$$\|g\|_{B_\delta} := \sum_{k\in\mathbb{Z}} |\widehat{g}(k)|\delta^{|k|}, \quad \delta > 1.$$

On the other hand, the stability can be false if one controls only finitely many derivates of the initial data.

Main result

Theorem (Han-Kwan - I., 2015)

Consider non-negative initial data of the form

$$f_{0,\varepsilon} = g_{0,\varepsilon} + h_{0,\varepsilon},$$

where $(g_{0,\varepsilon})$ *is a sequence of continuous functions satisfying*

$$\sup_{\varepsilon\in(0,1)} \sup_{v\in\mathbb{R}} (1+v^2)\|g_{0,\varepsilon}(\cdot,v)\|_{B_{\delta_0}} \leq C,$$

$$\sup_{\varepsilon\in(0,1)} \left\| \int_{\mathbb{R}} g_{0,\varepsilon}(\cdot,v)\,dv - 1 \right\|_{B_{\delta_0}} < \eta,$$

for some $\delta_0 > 1$*,* $C, \eta > 0$ *with* η *small enough.*

Theorem (continued...)

Assume that $h_{0,\varepsilon}$ are very small in the 1-Wasserstein distance:

$$\forall \varepsilon > 0, \quad W_1(h_{0,\varepsilon}, 0) \lesssim \exp(-\exp(C/\varepsilon^2)).$$

Then there exists $T > 0$ such that any weak solution of (VPME)$_\varepsilon$ starting from $f_{0,\varepsilon}$ converge to a solution of (KIE) starting from $g_0 = \lim_{\varepsilon \to 0} g_{0,\varepsilon}$ on $[0, T]$.

Comments

The smallness assumption on $h_{0,\varepsilon}$ is necessary.

Theorem (Grenier, 1999; Han-Kwan - Hauray, 2014.)

There exist smooth homogeneous equilibria $\mu(v)$ for (KIE) such that the following holds. For any $N > 0$ and $s > 0$, there exists a sequence of non-negative initial data $(f_{0,\varepsilon})$ such that

$$\|f_{\varepsilon,0} - \mu\|_{W^{s,1}_{x,v}} \leq \varepsilon^N,$$

but, for $\alpha \in [0,1)$,

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon^\alpha]} W_1(f_\varepsilon(t), \mu) > 0.$$

Strategy of the proof

(1) Prove quantitive weak-strong stability estimates for $(\mathrm{VPME})_\varepsilon$.
(2) Show that $f_\varepsilon(t)$ remains close to $g_\varepsilon(t)$, the solution of $(\mathrm{VPME})_\varepsilon$ starting from $g_{0,\varepsilon}$.
(3) Apply Grenier's result to $g_\varepsilon(t)$.

The key step is (1).

Proof of (1): splitting

Decompose E_ε as $\bar{E}_\varepsilon + \widehat{E}_\varepsilon$ where

$$\bar{E}_\varepsilon = -\bar{U}_\varepsilon', \qquad \widehat{E}_\varepsilon = -\widehat{U}_\varepsilon',$$

and $\bar{U}_\varepsilon$ and $\widehat{U}_\varepsilon$ solve respectively

$$\varepsilon^2 \bar{U}_\varepsilon'' = 1 - \rho_\varepsilon, \qquad \varepsilon^2 \widehat{U}_\varepsilon'' = e^{\bar{U}_\varepsilon + \widehat{U}_\varepsilon} - 1.$$

Then

$$(VPME)_\varepsilon := \begin{cases} \partial_t f_\varepsilon + v \cdot \partial_x f_\varepsilon + (\bar{E}_\varepsilon + \widehat{E}_\varepsilon) \cdot \partial_v f_\varepsilon = 0, \\ \bar{E}_\varepsilon = -\bar{U}_\varepsilon', \qquad \widehat{E}_\varepsilon = -\widehat{U}_\varepsilon', \\ \varepsilon^2 \bar{U}_\varepsilon'' = 1 - \rho_\varepsilon, \\ \varepsilon^2 \widehat{U}_\varepsilon'' = e^{\bar{U}_\varepsilon + \widehat{U}_\varepsilon} - 1, \\ f_\varepsilon(x, v, 0) \geq 0, \quad \int f_\varepsilon(x, v, 0)\, dx\, dv = 1. \end{cases}$$

Proof of (1): scaling

Define

$$\mathcal{F}_\varepsilon(t,x,v) := \frac{1}{\varepsilon} f_\varepsilon\left(\varepsilon t, x, \frac{v}{\varepsilon}\right).$$

Then

$$(VPME)_{\varepsilon,2} := \begin{cases} \partial_t \mathcal{F}_\varepsilon + v \cdot \partial_x \mathcal{F}_\varepsilon + (\bar{\mathcal{E}}_\varepsilon + \widehat{\mathcal{E}}_\varepsilon) \cdot \partial_v \mathcal{F}_\varepsilon = 0, \\ \bar{\mathcal{E}}_\varepsilon = -\bar{\mathcal{U}}_\varepsilon', \qquad \widehat{\mathcal{E}}_\varepsilon = -\widehat{\mathcal{U}}_\varepsilon', \\ \bar{\mathcal{U}}_\varepsilon'' = 1 - \varrho_\varepsilon, \\ \widehat{\mathcal{U}}_\varepsilon'' = e^{(\bar{\mathcal{U}}_\varepsilon + \widehat{\mathcal{U}}_\varepsilon)/\varepsilon^2} - 1, \\ \mathcal{F}_\varepsilon(x,v,0) \geq 0, \ \ \int \mathcal{F}_\varepsilon(x,v,0)\, dx\, dv = 1, \end{cases}$$

where

$$\varrho_\varepsilon(t,x) := \int \mathcal{F}_\varepsilon(t,x,v)\, dv.$$

Remark: $\bar{\mathcal{U}}_\varepsilon$ is just the classical Vlasov-Poisson potential, so

$$\bar{\mathcal{E}}_\varepsilon(t,x) = -\int_{\mathbb{T}} W'(x-y)\varrho_\varepsilon(t,y)\,dy,$$

where

$$W(x) := \frac{x^2 - |x|}{2}$$

(we are identifying $\mathbb{T}$ with $[-1/2, 1/2]$ with periodic boundary conditions).

In particular, $\bar{\mathcal{U}}_\varepsilon$ is 1-Lipschitz and $|\bar{\mathcal{U}}_\varepsilon| \leq 1$.

Proof of (1): Estimates for $\widehat{\mathcal{U}}$.

Using arguments from the calculus of variations, we show the following:

Lemma

There exists a unique solution of

$$\widehat{\mathcal{U}}'' = e^{(\bar{\mathcal{U}}+\widehat{\mathcal{U}})/\varepsilon^2} - 1 \qquad \textit{on } \mathbb{T}.$$

Also

$$\|\widehat{\mathcal{U}}\|_\infty \leq 3, \quad \|\widehat{\mathcal{U}}'\|_\infty \leq 2, \quad \|\widehat{\mathcal{U}}''\|_\infty \leq \frac{3}{\varepsilon^2}.$$

Idea of the proof.

We show that $\widehat{\mathcal{U}}$ is the unique minimizer of the functional

$$h \mapsto E[h] := \int_{\mathbb{T}} \left(\frac{1}{2}(h')^2 + \varepsilon^2 e^{(\bar{\mathcal{U}}+h)/\varepsilon^2} - h \right) dx$$

among all periodic functions $h : [-1/2, 1/2] \to \mathbb{R}$.

Then, using the Euler-Lagrange equations and the boundedness of the energy, we show the desired bounds. $\square$

Proof of (1): A Lagrangian approach

Consider $Z_0 = (X_0, V_0) : [0,1] \to \mathbb{R}^2$ a random variable with law $\mathcal{F}(0)$, that is

$$(Z_0)_\# ds = \mathcal{F}(0,x,v)\, dx\, dv.$$

Let $Z_t = (X_t, V_t)$ evolve as

$$\begin{cases} \dot{X}_t = V_t, \\ \dot{V}_t = \bar{\mathcal{E}}(X_t) + \widehat{\mathcal{E}}(X_t), \\ \bar{\mathcal{E}} = -\bar{\mathcal{U}}', \qquad \widehat{\mathcal{E}} = -\widehat{\mathcal{U}}', \\ \bar{\mathcal{U}}'' = 1 - \varrho, \\ \widehat{\mathcal{U}}'' = e^{(\bar{\mathcal{U}}+\widehat{\mathcal{U}})/\varepsilon^2} - 1, \\ \varrho(t) = (X_t)_\# ds. \end{cases}$$

Then the law $\mathcal{F}(t)$ of Z_t solves $(VPME)_{\varepsilon,2}$.

W_1 vs. L^1 distance

Recall:

$$W_1(\mu,\nu) = \min_{X_\# ds=\mu,\ Y_\# ds=\nu} \int_0^1 |X(s) - Y(s)|\, ds.$$

Let $\mathcal{F}^1, \mathcal{F}^2$ be two solutions of $(\mathrm{VPME})_{\varepsilon,2}$, and choose Z_0^1, Z_0^2 with law $\mathcal{F}^1(0), \mathcal{F}^2(0)$ such that

$$W_1(\mathcal{F}^1(0), \mathcal{F}^2(0)) = \int_0^1 |Z_0^1(s) - Z_0^2(s)|\, ds.$$

Then, since $W_1(\mathcal{F}^1(t), \mathcal{F}^2(t)) \le \int_0^1 |Z_t^1(s) - Z_t^2(s)|\, ds$, it is enough to estimate

$$\int_0^1 |Z_t^1(s) - Z_t^2(s)|\, ds.$$

We compute

$$\frac{d}{dt}\int_0^1 |Z_t^1(s) - Z_t^2(s)|\, ds.$$

Difficult terms:

$$\int_0^1 |\bar{\mathcal{E}}_t^1(X_t^1) - \bar{\mathcal{E}}_t^2(X_t^2)|\, ds$$

and

$$\int_0^1 |\widehat{\mathcal{E}}_t^1(X_t^1) - \widehat{\mathcal{E}}_t^2(X_t^2)|\, ds.$$

The first term can be controlled using the explicit form of $\bar{\mathcal{E}}$ (Hauray, 2012):

$$\int_0^1 |\bar{\mathcal{E}}_t^1(X_t^1) - \bar{\mathcal{E}}_t^2(X_t^2)|\, ds \leq 8\|\varrho^1(t)\|_\infty \int_0^1 |Z_t^1(s) - Z_t^2(s)|\, ds.$$

For the second term we need to carefully use the equation for $\widehat{\mathcal{U}}$ and its regularity properties to control $\int_0^1 |\widehat{\mathcal{E}}_t^1 - \widehat{\mathcal{E}}_t^2|$ in terms of $\int_0^1 |Z_t^1 - Z_t^2|$.

In the end, we prove

$$\frac{d}{dt}\int_0^1 |Z_t^1 - Z_t^2|\, ds \leq$$
$$\left(1 + 8\|\varrho^1(t)\|_\infty + \frac{3}{\varepsilon^2} + \frac{1}{\varepsilon^2} e^{15/(2\varepsilon^2)} \|\varrho^1(t)\|_\infty\right) \int_0^1 |Z_t^1 - Z_t^2|\, ds,$$

that yields

$$W_1(\mathcal{F}^1(t), \mathcal{F}^2(t)) \leq$$
$$e^{(1+3/\varepsilon^2)t + (8 + \frac{1}{\varepsilon^2} e^{15/(2\varepsilon^2)}) \int_0^t \|\varrho^1(\tau)\|_\infty \, d\tau}\, W_1(\mathcal{F}^1(0), \mathcal{F}^2(0)).$$

Proof of (1): an anisotropic W_1 distance

To go back to the original system, we need to relate $W_1(\mathcal{F}^1(t), \mathcal{F}^2(t))$ to $W_1(f^1(t), f^2(t))$.

We prove that

$$\varepsilon W_1(f^1(\varepsilon t), f^2(\varepsilon t)) \leq W_1(\mathcal{F}^1(t), \mathcal{F}^2(t)) \leq W_1(f^1(\varepsilon t), f^2(\varepsilon t)),$$

and we conclude

$$W_1(f^1(t), f^2(t)) \\ \leq \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon}\left[(1+3/\varepsilon^2)t + (8+\frac{1}{\varepsilon^2} e^{15/(2\varepsilon^2)}) \int_0^t \|\rho^1(\tau)\|_\infty \, d\tau\right]} W_1(f^1(0), f^2(0)).$$

Remark: For the classical Vlasov-Poisson system, our argument shows that

$$W_1(f^1(t), f^2(t)) \leq \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon}\left[t+8\int_0^t \|\rho^1(\tau)\|_\infty \, d\tau\right]} W_1(f^1(0), f^2(0)).$$

Quasineutral limit in higher dimensions

The higher dimensional case is more complicated because of worse estimates for the Poisson equation.

In this case we consider the classical Vlasov-Poisson equation

$$(VP)_\varepsilon := \begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x U_\varepsilon, \\ -\varepsilon^2 \Delta_x U_\varepsilon = \rho_\varepsilon - 1, \\ f_\varepsilon|_{t=0} = f_{0,\varepsilon} \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f_{0,\varepsilon}\, dx\, dv = 1, \end{cases}$$

The energy of this system:

$$\mathcal{E}(f_\varepsilon(t)) := \frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f_\varepsilon |v|^2 \, dv \, dx + \frac{\varepsilon^2}{2}\int_{\mathbb{T}^d} |\nabla_x U_\varepsilon|^2 \, dx.$$

Its formal limit:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f = 0, \\ E = -\nabla_x U, \\ \rho = 1, \\ f|_{t=0} = f_0 \geq 0, \ \ \int_{\mathbb{T}^d\times\mathbb{R}^d} f_0 \, dx \, dv = 1. \end{cases}$$

Main result

Theorem (Han-Kwan - I., 2015)

Let $d = 2, 3$. The convergence holds as in the 1D case provided, in addition, there exist $C_0, \gamma > 0$ such that:

- *(uniform estimates)*

$$\|f_{0,\varepsilon}\|_\infty \leq C_0, \quad \mathcal{E}(f_{0,\varepsilon}) \leq C_0.$$

- *(compact support in velocity)*

$$f_{0,\varepsilon}(x, v) = 0 \quad \textit{if } |v| > \frac{1}{\varepsilon^\gamma}.$$

- *We add some correctors taking care of the "plasma oscillations".*

Strategy of the proof

(1) We show quantitive strong-strong stability estimates à la Loeper for $(\mathrm{VP})_\varepsilon$.

More precisely, up to suitable correctors, we control $W_2(f_\varepsilon(t), g_\varepsilon(t))$ in terms of $W_2(f_\varepsilon(0), g_\varepsilon(0))$ and the L^∞ norm of $\int f_\varepsilon\, dv$ and $\int g_\varepsilon\, dv$.

(2) To bound the L^∞ norm of $\int f_\varepsilon\, dv$, we control the growth of the support in velocity of f_ε.

(3) We conclude as in the 1D case, by applying Grenier's result.

THAT'S ALL!!

Thanks for your attention!